NOAO ADVANCED DEVELOPMENT PROGRAM

R&D Note

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SUBJECT: Curvature sensing: a diffraction theory

1. Optical set-up

The optical set-up is is schematically described in Fig. 1. A distorted plane wave is focussed by a lens or a mirror L_1, with focal length f, on its focal plane F. The curvature sensor consists of two image detectors. One detect the irradiance distribution in plane P_1 at a distance l before F. The other detect the irradiance distribution in plane P_2 at the same distance l after F. For the sake of symmetry, a second lens L_2 of focal length $f/2$ is used in plane F to reimage L_1 at a distance f beyond F. It is shown that, at the geometrical approximation, the difference between the irradiance distributions in planes P_1 and P_2 is a measure of the local wavefront curvature and of the wavefront radial tilt at the edge of the pupil.

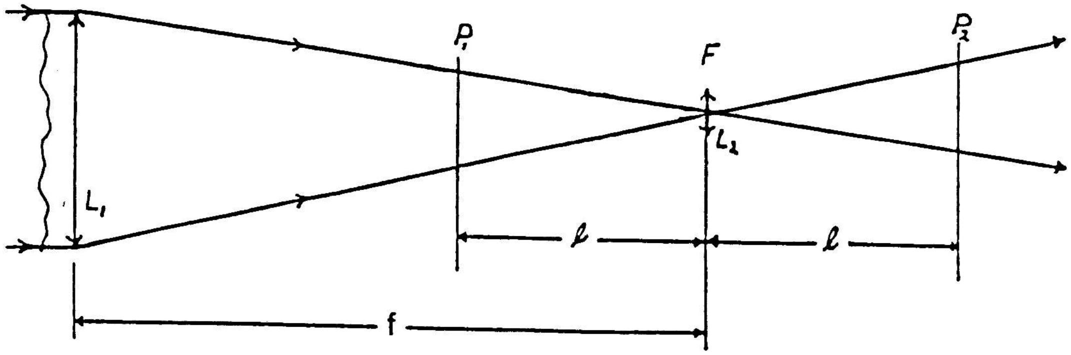

Figure 1

2. Irradiance distribution in plane P_1

Let $\Psi(\vec{r})$ be the complex amplitude of the incoming wavefront, and $P(\vec{r})$ the pupil transmission function (equal to one inside the pupil and equal to zero outside). The complex amplitude at the lens (or mirror) output is

$$A_0(\vec{r}) = P(\vec{r})\Psi(\vec{r})\exp -i\pi\frac{\vec{r}^2}{\lambda f} \tag{1}$$

The complex amplitude diffracted in plane P_1 is given by [1]

$$A_1(\vec{r}) = P(\vec{r})\Psi(\vec{r})\exp -i\pi\frac{\vec{r}^2}{\lambda f} * \frac{1}{i\lambda(f-l)}\exp i\pi\frac{\vec{r}^2}{\lambda(f-l)} \tag{2}$$

where the convolution operator $*$ describes Fresnel diffraction over a distance $f-l$. Expressing the convolution as an integral gives

$$A_1(\vec{r}) = \frac{1}{i\lambda(f-l)} \int P(\vec{r}\,')\Psi(\vec{r}\,')\exp -i\pi\frac{\vec{r}\,'^2}{\lambda f}\exp i\pi\frac{(\vec{r}-\vec{r}\,')^2}{\lambda(f-l)}d\vec{r}\,'$$
$$= \frac{1}{i\lambda(f-l)}\exp i\pi\frac{\vec{r}^{\,2}}{\lambda(f-l)}\int P(\vec{r}\,')\Psi(\vec{r}\,')\exp i\pi\frac{l\vec{r}\,'^2}{\lambda f(f-l)}$$
$$\times \exp -2i\pi\frac{\vec{r}\,\vec{r}\,'}{\lambda(f-l)}d\vec{r}\,' \qquad (3)$$

The related irradiance distribution is the square of the complex amplitude

$$I_1(\vec{r}) = |A(\vec{r})|^2$$
$$= \frac{1}{\lambda^2(f-l)^2}\int\int P(\vec{r}\,')P(\vec{r}\,'')\Psi(\vec{r}\,')\Psi^*(\vec{r}\,'')$$
$$\times \exp i\pi\frac{l(\vec{r}\,'^2-\vec{r}\,''^2)}{\lambda f(f-l)}\exp 2i\pi\frac{\vec{r}(\vec{r}\,''-\vec{r}\,')}{\lambda(f-l)}d\vec{r}\,'d\vec{r}\,'' \qquad (4)$$

or, changing variable $\vec{r}\,''$ into $\vec{r}\,'+\vec{\rho}$, Eq. (4) becomes

$$I_1(\vec{r}) = \frac{1}{\lambda^2(f-l)^2}\int \exp -i\pi\frac{l\vec{\rho}^{\,2}}{\lambda f(f-l)}\exp 2i\pi\frac{\vec{\rho}\,\vec{r}}{\lambda(f-l)}$$
$$\times \int P(\vec{r}\,')P(\vec{r}\,'+\vec{\rho})\Psi(\vec{r}\,')\Psi(\vec{r}\,'+\vec{\rho})\exp -2i\pi\frac{l\vec{\rho}\,\vec{r}\,'}{\lambda f(f-l)}d\vec{r}\,'d\vec{\rho} \qquad (5)$$

3. Irradiance distribution in plane P_2

The complex amplitude $A_F(\vec{r})$ diffracted in plane F is obtained by setting l equal to zero in Eq. (3)

$$A_F(\vec{r}) = \frac{1}{i\lambda f}\exp i\pi\frac{\vec{r}^{\,2}}{\lambda f}\int P(\vec{r}\,')\Psi(\vec{r}\,')\exp -2i\pi\frac{\vec{r}\,\vec{r}\,'}{\lambda f}d\vec{r}\,' \qquad (6)$$

or, after transmission through lens L_2,

$$A'_F(\vec{r}) = \frac{1}{i\lambda f}\exp -i\pi\frac{\vec{r}^{\,2}}{\lambda f}\int P(\vec{r}\,')\Psi(\vec{r}\,')\exp -2i\pi\frac{\vec{r}\,\vec{r}\,'}{\lambda f}d\vec{r}\,' \qquad (7)$$

The complex amplitude $A_2(\vec{r})$ diffracted further over a distance l is obtained by convolving again with the Fresnel diffraction operator

$$A_2(\vec{r}) = A'_F(\vec{r}) * \frac{1}{i\lambda l}\exp i\pi\frac{\vec{r}^{\,2}}{\lambda l} \qquad (8)$$

which gives after a few manipulations

$$A_2(\vec{r}) = \frac{1}{i\lambda(f-l)}\exp -i\pi\frac{\vec{r}^{\,2}}{\lambda(f-l)}\int P(\vec{r}\,')\Psi(\vec{r}\,')\exp -i\pi\frac{l\vec{r}\,'^2}{\lambda f(f-l)}$$
$$\times \exp -2i\pi\frac{\vec{r}\,\vec{r}\,'}{\lambda(f-l)}d\vec{r}\,' \qquad (9)$$

a relation very similar to (3). The related irradiance distribution is very similar to (5)

$$I_2(\vec{r}) = \frac{1}{\lambda^2(f-l)^2}\int \exp i\pi\frac{l\vec{\rho}^{\,2}}{\lambda f(f-l)}\exp 2i\pi\frac{\vec{\rho}\,\vec{r}}{\lambda(f-l)}$$

$$\times \int P(\vec{r}')P(\vec{r}'+\vec{\rho})\Psi(\vec{r}')\Psi(\vec{r}'+\vec{\rho}) \exp 2i\pi \frac{l\vec{\rho}\,\vec{r}'}{\lambda f(f-l)} d\vec{r}' d\vec{\rho} \tag{10}$$

3. The geometrical optics approximation

Let ρ_0 be the correlation length of the complex amplitude disturbance of the incoming wavefront. Physically, wavefront fluctuations of scale ρ_0 diffract light over an angle λ/ρ_0 and produce on plane P_1 a blur of size $\lambda(f-l)/\rho_0$. This blur must be small compared to the size of the fluctuations we want to measure which is ρ_0 scaled down by a factor l/f, i.e. one must have

$$\frac{\lambda(f-l)}{\rho_0} \ll \frac{\rho_0 l}{f} \qquad \text{or,} \qquad \frac{\lambda f(f-l)}{l\rho_0^2} \ll 1 \tag{11}$$

We assume that this condition is met in the following.

The contribution to the integral over $\vec{r}'$ in Eqs (5) or (10) takes significant values only when $\lambda f(f-l)/l|\vec{\rho}|$ is larger than or of the order of ρ_0, i.e. when

$$|\vec{\rho}| \leq \frac{\lambda f(f-l)}{\rho_0 l} \tag{12}$$

When condition (11) is met this occurs when $|\vec{\rho}| \ll \rho_0$, hence the product $\Psi(\vec{r}')\Psi(\vec{r}'+\vec{\rho})$ in Eqs (5) and (10) can be replaced by an approximate expression valid only when $|\vec{\rho}| \ll \rho_0$. Assuming pure phase disturbances and calling $\phi(\vec{r}')$ the phase of the incoming wavefront, we have

$$\Psi(\vec{r}')\Psi^*(\vec{r}'+\vec{\rho}) = \exp -i\left(\phi(\vec{r}'+\vec{\rho}) - \phi(\vec{r}')\right) \tag{13}$$

or, for $|\vec{\rho}| \ll \rho_0$,

$$\Psi(\vec{r}')\Psi^*(\vec{r}'+\vec{\rho}) \approx \exp -i\vec{\rho}\nabla\phi(\vec{r}')$$
$$\approx 1 - i\vec{\rho}\nabla\phi(\vec{r}') \tag{14}$$

Since ρ_0 is much smaller than the entrance pupil, under the same conditions, we also have

$$P(\vec{r}')P(\vec{r}'+\vec{\rho}) \approx P(\vec{r}') \tag{15}$$

Hence, Eq. (5) becomes

$$I_1(\vec{r}) = \frac{1}{\lambda^2(f-l)^2} \int \exp -i\pi \frac{l\vec{\rho}^2}{\lambda f(f-l)} \exp 2i\pi \frac{\vec{\rho}\,\vec{r}}{\lambda(f-l)}$$
$$\times \int P(\vec{r}')\left(1 - i\vec{\rho}\nabla\phi(\vec{r}')\right) \exp -2i\pi \frac{l\vec{\rho}\,\vec{r}'}{\lambda f(f-l)} d\vec{r}' d\vec{\rho} \tag{16}$$

The second integral in Eq. (16) is given by the Fourier transform of $P(\vec{r}')\left(1 - i\vec{\rho}\nabla\phi(\vec{r}')\right)$ which becomes negligibly small at frequencies larger than $1/\rho_0$ Hence this integral takes significant values only for

$$\frac{l|\vec{\rho}|}{\lambda f(f-l)} \leq \frac{1}{\rho_0} \qquad \text{or,} \qquad |\vec{\rho}| \leq \frac{\lambda f(f-l)}{l\rho_0} \tag{17}$$

in which case, according to (11), the quadratic phase in the first integral becomes very small compared to unity

$$\frac{l\vec{\rho}^2}{\lambda f(f-l)} \leq \frac{\lambda f(f-l)}{l\rho_0^2} \ll 1$$

and can be neglected, yielding

$$I_1(\vec{r}) = \frac{1}{\lambda^2(f-l)^2}\int \exp 2i\pi\frac{\vec{\rho}\,\vec{r}}{\lambda(f-l)} \times \int P(\vec{r}')\left(1 - i\vec{\rho}\nabla\phi(\vec{r}')\right)\exp -2i\pi\frac{l\vec{\rho}\,\vec{r}'}{\lambda f(f-l)}\mathrm{d}\vec{r}'\mathrm{d}\vec{\rho} \tag{18}$$

Similarly, Eq. (10) gives

$$I_2(\vec{r}) = \frac{1}{\lambda^2(f-l)^2}\int \exp 2i\pi\frac{\vec{\rho}\,\vec{r}}{\lambda(f-l)} \times \int P(\vec{r}')\left(1 - i\vec{\rho}\nabla\phi(\vec{r}')\right)\exp 2i\pi\frac{l\vec{\rho}\,\vec{r}'}{\lambda f(f-l)}\mathrm{d}\vec{r}'\mathrm{d}\vec{\rho} \tag{19}$$

The approximation we have made is called the geometrical approximation since it yields results predicted by geometrical optics. It was used by Reiger in his early theory of stellar scintillation [2].

4. The sensor signal

When the incoming wavefront is a plane wave (ϕ=0), Eqs (18) gives

$$I(\vec{r}) = \frac{1}{\lambda^2(f-l)^2}\int \exp 2i\pi\frac{\vec{\rho}\,\vec{r}}{\lambda(f-l)}P(\vec{r}')\exp 2i\pi\frac{l\vec{\rho}\,\vec{r}'}{\lambda f(f-l)}\mathrm{d}\vec{r}'\mathrm{d}\vec{\rho} = (f/l)^2P(f\vec{r}/l) \tag{20}$$

i.e. the illumination is uniform over a reduced size pupil "image" with brightness $(f/l)^2$ times the entrance pupil brightness as expected from geometrical optics. Similarly Eq. (19) gives

$$I(\vec{r}) = (f/l)^2P(-f\vec{r}/l) \tag{21}$$

describing an inverted but otherwise identical pupil image. When the incoming beam is distorted, the illumination $I_1(\vec{r})$ in plane P_1 given by Eq. (18) is the sum of the above uniform illumination plus a fluctuation

$$\Delta I_1(\vec{r}) = \frac{-i}{\lambda^2(f-l)^2}\int \exp 2i\pi\frac{\vec{\rho}\,\vec{r}}{\lambda(f-l)} \times \vec{\rho}\cdot\int P(\vec{r}')\nabla\phi(\vec{r}')\exp -2i\pi\frac{l\vec{\rho}\,\vec{r}'}{\lambda f(f-l)}\mathrm{d}\vec{r}'\mathrm{d}\vec{\rho} \tag{22}$$

Changing $\vec{\rho}$ into

$$\vec{\rho} = \frac{\lambda f(f-l)}{l}\vec{u},$$

gives

$$\Delta I_1(\vec{r}) = \frac{-i\lambda f^3(f-l)}{l^3}\int \exp 2i\pi\frac{f}{l}\vec{u}\vec{r} \times \vec{u}\cdot\int P(\vec{r}')\nabla\phi(\vec{r}')\exp -2i\pi\vec{u}\vec{r}'\,\mathrm{d}\vec{r}'\mathrm{d}\vec{u} \tag{23}$$

or,

$$
\begin{aligned}
\Delta I_1(\vec{r}) &= \frac{-\lambda f^3(f-l)}{2\pi l^3} \nabla \left[P(f\vec{r}/l) \nabla \phi(f\vec{r}/l) \right] \\
&= \frac{-\lambda f^3(f-l)}{2\pi l^3} \left[\frac{\partial}{\partial \vec{n}} \phi(f\vec{r}/l) \delta_c + \nabla^2 \phi(f\vec{r}/l) \right]
\end{aligned}
\tag{24}
$$

where δ_c represents a linear impulse distribution around the pupil edge. The derivative $\partial\phi/\partial\vec{n}$ is the radial wavefront tilt at the pupil edge.

Similarily, Eq. (19) describes the illumination $I_2(\vec{r})$ in plane P_2 as a sum of a uniform illumination $I(\vec{r})$ given by Eq. (21) plus a fluctuation

$$
\Delta I_2(\vec{r}) = \frac{\lambda f^3(f-l)}{2\pi l^3} \left[\frac{\partial}{\partial \vec{n}} \phi(-f\vec{r}/l) \delta_c + \nabla^2 \phi(-f\vec{r}/l) \right]
\tag{25}
$$

We take as the sensor signal $S(\vec{r})$ the difference $I_2(\vec{r}) - I_1(-\vec{r})$ between the illuminations in planes P_1 and P_2 normalized with $I(\vec{r})$ or, using Eqs (20), (24) and (25)

$$
\begin{aligned}
S(\vec{r}) &= \frac{\Delta I_2(\vec{r}) - \Delta I_1(-\vec{r})}{I(\vec{r})} \\
&= \frac{-\lambda f(f-l)}{\pi l} \left[\frac{\partial}{\partial \vec{n}} \phi(f\vec{r}/l) \delta_c + \nabla^2 \phi(f\vec{r}/l) \right]
\end{aligned}
\tag{27}
$$

Eq. (27) shows that the sensor will map the wavefront curvature $\nabla^2\phi$ on the pupil and the wavefront radial tilt $\partial\phi/\partial\vec{n}$ at the edge. This information allows complete wavefront retrieval by solving the Poisson equation with the observed boundary conditions.

5. References

[1] Goodman, J.W., *Introduction to Fourier Optics*, McGraw Hill publ. (1968).

[2] Reiger, S.H., *Astronomical Journal*, Vol. **68**, p. 295 (1963).