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ON THE GROWTH OF CONDENSATIONS IN AN EXPANDING UNIVERSE

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ABSTRACT

Most of this article concerns the growth of small amplitude fluctuations in an evolving Friedmann universe, after the perturbation is inside the particle horizon and before the gas has recombined. The matter and radiation field interaction is treated by introducing a 4-vector whose components give the transfer of energy and momentum. The momentum exchange is obtained from the radiation transfer equation, and the expression for this stress is valid as long as the fluctuation is not too transparent. The main results are the following: Perturbation masses $> 10^{16}\ M_\odot$ will not oscillate prior to the cosmological recombination period. Smaller masses do oscillate for some time, but there comes an era when the radiation and matter fields start to slip past each other and the oscillations are damped out. This occurs well before recombination and while the perturbation is still opaque. (The damping can begin even when the optical depth is of the order of a few hundred.)

A simple non-linear treatment of the flow after recombination is presented. When the gas goes neutral we find density and/or velocity amplitudes $\sim 1\%$ are needed in order to have the motions reverse around 10^9 years. This amplitude requirement can now be traced back to a much earlier time. After the perturbation lies within the particle horizon the growth rate is independent of the choice of particular coordinates. This study then shows that at this early epoch the perturbation amplitudes must be around one percent — a stringent requirement.

I. INTRODUCTION

For purposes of discussion in this article we define the following five regimes during the development of a perturbation in an evolving Friedmann universe.

I. If we consider perturbations which encompass enough matter to be of astrophysical interest, and if we assume such perturbations were present when the universe was very young, then they must have extended well beyond the particle horizon. During this time gravity is dominant over pressure gradient forces.

II. After the patch of higher density has come inside the horizon, pressure gradient forces are dominant and the system starts to oscillate. The first, as well as part of the second phase, has been studied by Lifshitz (1946), Lifshitz and Khalatnikov (1963), Hawking (1966), and Sachs and Wolfe (1967). Although it would appear they have worked the same physical problem, some got different solutions for the growing and damping modes. Evidently this is due to different choices for the space-like hypersurfaces;* and, conse-

* The author of this paper died on March 27, 1969, and the manuscript was not in the final form that he would have preferred. It was first submitted to the *Astrophysical Journal* on September 1, 1967. Although Dr. Michie made some minor modifications in response to a referee's suggestions, prolonged illness prevented his making the more extensive alterations and additions that would have properly related this work to the recent important paper by Silk (1968). — *J. W. Chamberlain*

* I am indebted to Dr. James Bardeen for some very illuminating discussions concerning these solutions.

quently, until the patch is inside the horizon the time dependence of the development will be coordinate dependent.

III. All the solutions mentioned have been for an adiabatic flow. As will be shown in this article, there may come a later phase when the time scale for photon diffusion across the patch is less than the cosmic expansion time. When this occurs, the radiation and matter fields will start to slip and the oscillations will be damped. After the damping we will find a period of slow gravitational growth. Throughout this article when we refer to growth, we mean an increase in $\Delta\rho/\rho$ and not to an increase in, e.g., size. Of course if small enough, the patch may later become quite transparent while the gas is still fully ionized. Peebles (1965) has found the radiation field still exerts an enormous force on the matter and thus slows down the growth rate.

IV. The cosmic gas starts to recombine when the universe is around 10^5 to 10^6 years old (depending on what one adopts for the value of the present radiation temperature and mass density). Large-scale perturbations may enter this period still opaque, but it appears there is no new or enhanced instability (Saslaw, 1967).

V. A fully Newtonian phase occurs after recombination, and this has been studied by Bonnor (1957). He found for the growing mode the density amplitude increases at $t^{2/3}$ as long as the flow is linear.

The solutions obtained in this article are valid only after the patch is inside the particle horizon, and only after nuclear reactions have ceased. Part of phase II, all of III, and in some cases part of phase IV are the periods of applicability. The solutions are of value for the following reasons: they largely fill the gap between the Lifshitz and Bonnor solutions; they constitute the solutions for the earliest epoch which are not coordinate dependent; they indicate to what extent the development depends on the present value of the cosmic radiation temperature (we use 0.3 and 3.0 degrees Kelvin); they determine when the flow is not adiabatic, how much the oscillations are damped, and how long it takes to achieve the damping; and lastly, when combined with a nonlinear calculation for the fifth phase development, the solutions will give the information needed to answer the question: When a perturbation comes within the horizon, what minimum amplitude is required to later obtain a stellar system when the universe is a few 10^9 years old?

The chemical composition of the gas is assumed to be hydrogen only. Except when stated otherwise, all quantities are expressed in c.g.s. units.

II. THE BASIC EQUATIONS

It will be convenient at the outset to list for reference certain approximations to be used in the derivations to follow.

A_1: Within a spherical region of radius l, part of the interior gravitational mass is $\overline{M}$ arising from the mean field values for ρ and P. The mass arising from the perturbation is $\delta M = M - \overline{M}$, and we require

$$G\,|\,\delta M\,|\,/\,lc^2 << 1.$$

(G is the gravitational constant and c is the speed of light.) The coordinates will be comoving with the mean *total* field and they are extended into the perturbed region. The above approximation ensures there will be little deviation of the "true" coordinates from the mean field comoving coordinates.

A_2: We will assume the velocity of the matter (V_m) in the perturbed region relative to these coordinates is small, so that terms of order $(V_m\,/\,c)^2$ can be neglected.

A_3: The only source of the gravitational field acting on a particular perturbation will be from the perturbation itself. Thus tidal forces due to outside fluctuations will be ignored.

Instead of perturbing all the components of the metric tensor we shall choose at the outset a particular point and consider a spherically symmetric region about this point, with motions only in the radial direction. Without loss of generality we may suppose the metric to be of the form

$$ds^2 = e^{\nu}c^2dt^2 - S^2(t)e^{\mu}\left[dr^2 + r^2(\sin^2\theta\, d\phi^2 + d\theta^2)\right] \quad (1)$$

where μ and ν are functions of r and t only. A_1 means both $|\mu|$ and $|\nu|$ are very small compared to 1. When applied to the mean field $\overline{\mu} = 0 = \overline{\nu}$, and the line element is the well-known Friedmann type derived for comoving coordinates; the curvature is zero.

A_4: For the final calculations, time derivatives of μ and ν will be neglected. This approximation is valid only after the patch lies within the particle horizon.

a) The Gravitational Force.

The μ and ν functions are obtained from the general relativity field equations

$$R^i{}_j = -8\pi\frac{G}{c^4}\left(T^i{}_j - \frac{1}{2}\eta^i{}_j T\right) \quad (2)$$

where $R^i{}_j$ is the Ricci tensor and $T^i{}_j$ is the stress-energy tensor of the medium considered (in this paper the convention will be adopted of letting Latin indices take the values 0, 1, 2, and 3, and the Greek indices take only the values 1, 2, and 3 referring to the spatial coordinates; also the summation over repeated indices will be restricted to their respective ranges). Also T is the contracted $T^i{}_j$, and the components of $\eta^i{}_j$ are 1 along the principal diagonal, and zero otherwise.

We will take for the form of the stress-energy tensor

$$T^i{}_j = (\epsilon + P)\, U^i U_j - \eta^i{}_j P \quad (3)$$

with ϵ and P the proper values of the total energy density (including rest mass energy) and pressure, respectively. The 4-velocity U^i equals dx^i / ds with the identification $dx^0 = dt$.

Not all the components of $R^i{}_j$ are independent since $T^i{}_j$ must satisfy the condition

$$T^i{}_{j;i} = 0. \quad (4)$$

Covariant differentiation is indicated by the semicolon.

The solution for μ and ν is now straightforward, and only an outline of the method need be presented. In view of the approximation A_1, since μ and ν are of the order $G|\delta M|/lc^2$, products of μ, ν and their derivatives may be neglected. This means we seek a solution linear in G. All components of $R^i{}_j$ may now be written down, linearized with respect to μ and ν, and the terms referring solely to the mean field subtracted out.

From (3) and A_2

$$R^2{}_2 - R^1{}_1 = 0 \quad (5)$$

and it is found that this means

$$\mu + \nu = 0. \quad (6)$$

From the $R^0{}_1$ equation there results

$$\frac{\partial \mu}{\partial t} = \nu\frac{\dot{S}}{S} - f \quad (7)$$

where

$$f = 8\pi\frac{G}{c^2}\int e^\nu\, T^0{}_1\, dr. \quad (8)$$

Finally, by first substituting the right side of equation (7) for the μ and ν time derivatives and then introducting a potential ϕ $(= c^2\,\nu/2)$, one can write from the $R^0{}_0$ equation, using proper distances for space derivatives,

$$\nabla^2\phi = 4\pi G\left(\frac{\delta\epsilon + 3\delta P}{c^2}\right) - 12\pi\frac{G}{c^2}S\int_r^\infty\left\{\left[\frac{\partial}{\partial t}\left(\rho_m V_m\right) + 3\mathrm{H}\rho_m V_m\right] + \left[\frac{\partial}{\partial t}\left(\frac{4}{3}\rho_r V_r\right) + 4\mathrm{H}\rho_r V_r\right]\right\}dr. \quad (9)$$

In this expression ρ_m and ρ_r are the mass density of matter and radiation respectively, H equals $\dot{S}/S$, and V_r is defined analagous to V_m: it is that velocity an observer must have in order to measure a zero value for the net radiation flux.

The above expression assumes the simpler form

$$\nabla^2\phi = 4\pi G\left(\frac{\delta\epsilon + 3\,\delta P}{c^2}\right) \quad (10)$$

when the patch is inside the particle horizon, for if l_{H} $(= c/\mathrm{H})$ is the distance to this horizon and λ is the wavelength of the perturbation, then the integral is order $(\lambda/l_{\mathrm{H}})^2$ times smaller than the leading term on the right side of equation (9). Approximation A_3 has also been used, and hence all four are needed to obtain the Poisson equation (10). It can be shown that equation (10) would produce too strong a gravitational force if used for phase I studies.

b) The Conservation Equations

To obtain the appropriate post-Newtonian fluid equations for the matter, the tensor T^{ij} is written as a sum of the matter and radiation tensors

$$T^{ij} = T^{ij}{}_{[m]} + T^{ij}{}_{[r]}, \quad (11)$$

and to satisfy the divergence condition (equation 4) we introduce a 4-vector Q^i by writing

$$T^{ij}{}_{[m];j} = -Q^i \quad (12a)$$

and

$$T^{ij}{}_{[r];j} = +Q^i \quad (12b)$$

This vector represents the exchange of energy and momentum between the radiation and matter. We must also ensure the conservation of baryon number, and to do this we use

$$(\rho_m U^i)_{;i} = 0. \qquad (13)$$

Equation (12a) gives the momentum and energy conservation equations for the matter field when we put i equal to 1 and 0, respectively. Finally, there must be added the equation for the transfer of radiation; this is written in section d) where we solve it for the radiation stress and flux.

At this point the role of approximation A_4 must be clearly understood. During the second phase the gravitational force is negligible compared to pressure forces, and hence the approximations used to obtain equation (10) produce little error. But time derivatives of μ and ν also enter into the above energy and baryon conservation equations. Thus during the beginning of the second phase a numerical solution would be in some error if $\partial\mu/\partial t$ and $\partial\nu/\partial t$ were neglected. It is for this reason we emphasize that our solutions presented in section III are accurate only *after* the system has undergone a few oscillations, for then these time derivatives will have no effect.

c) The Mean Field Equations

The first of the two mean field equations is for the expansion parameter H $(= \dot{S}/S)$:

$$\mathrm{H}^2 = \frac{8}{3}\pi G\bar{\rho}_m(t)\left[1 + \alpha_o\left(\frac{\bar{\rho}_m(t)}{\bar{\rho}_m(t_P)}\right)^{1/3}\right] \qquad (14)$$

where

$$\alpha_o = \left(\frac{\bar{\rho}_r}{\bar{\rho}_m}\right)_P = \left(\frac{a\bar{T}_r^{\,4}}{c^2\bar{\rho}_m}\right)_P. \qquad (15)$$

The subscript (P) means the quantity is to be evaluated at the present epoch. The other equation, giving the energy balance, is

$$\frac{\partial\bar{\epsilon}}{\partial t} = -\,3\mathrm{H}\,(\bar{\epsilon} + \bar{P}) - \bar{Q}_o \qquad (16)$$

Now the differential equations for the mean field variables are easily obtained. (We will drop the bar over these variables since by context there should be no confusion.) First, we may neglect excitation energy and write

$$\epsilon_m = \rho_m\,(c^2 + U_m), \qquad (17)$$

with the internal energy per gram

$$U_m = \frac{3}{2}\frac{P_m}{\rho_m} + \frac{\chi_{\mathrm{H}}}{m_{\mathrm{H}}}\left(\frac{\psi}{1+\psi}\right) \qquad (18)$$

being the sum of the thermal and ionization energies. χ_{H} is the ionization energy per hydrogen atom (mass m_{H}), and ψ which specifies the state of ionization is defined by

$$\psi = n_i/n_{\mathrm{H}}\,, \qquad (19)$$

the ratio of the ionized to the neutral particle densities. The equation of state is

$$P_m = \frac{\rho_m\,k\,T_e}{\mu\,m_{\mathrm{H}}} \qquad (20)$$

where

$$\mu = \left(\frac{1+\psi}{1+2\psi}\right) \qquad (21)$$

is the mean particle weight in terms of the hydrogen mass. We will also use the quantity β_e defined by

$$\beta_e = \frac{\chi_{\mathrm{H}}}{kT_e} = 1.58\times 10^5/T_e \qquad (22)$$

and β_r is defined in an analogous way using the radiation temperature.

It is an easy matter to obtain from equations (16) and (13),

$$\frac{1}{T_e}\frac{\partial T_e}{\partial t} = -2\mathrm{H} - \frac{2}{3}\frac{Q_o}{P_m} - \left(1 + \frac{2}{3}\beta_e\right)\frac{1}{(1+\psi)\,(1+2\psi)}\frac{\partial\psi}{\partial t} \qquad (23)$$

$$\frac{1}{\rho_m}\frac{\partial\rho_m}{\partial t} = -3\mathrm{H} \qquad (24)$$

$$\frac{1}{P_m}\frac{\partial P_m}{\partial t} = -5\mathrm{H} - \frac{2}{3}\frac{Q_o}{P_m} - \frac{2}{3}\beta_e\frac{1}{(1+\psi)\,(1+2\psi)}\frac{\partial\psi}{\partial t} \qquad (25)$$

for the matter field.

Next we may apply equation (16) to the radiation field by changing the sign before Q_o on the right side to (+) and using

$$P_r = \frac{1}{3}\epsilon_r = \frac{1}{3}\rho_r c^2, \qquad (26)$$

to get

$$\frac{1}{T_r}\frac{\partial T_r}{\partial t} = \frac{1}{4\rho_r}\frac{\partial\rho_r}{\partial t} = \frac{1}{4P_r}\frac{\partial P_r}{\partial t} = -\mathrm{H} + \frac{Q_o}{12P_r}\,. \qquad (27)$$

The dominant mechanism for energy exchange between the two fields is Compton (and inverse

Compton) scattering with free-free and free-bound transitions becoming important when $\rho_r \sim \rho_m$ or when the gas starts to recombine. This exchange of energy is so effective that the electron and radiation temperatures will be virtually the same before and during most of the recombination period. In all the calculations we will present, $|\delta| = |(T_e - T_r)/T_e|$ is less than 10^{-3}; T_r and T_e start to become significantly different only when ψ is less than $\sim 10^{-5}$ (see Weymann, 1966).

When the gas is fully ionized and even while partially ionized we may equate the left sides of equations (23) and (27) to obtain

$$Q_o = \frac{\left[H + \dfrac{\left(1+\frac{2}{3}\beta_e\right)}{(1+\psi)(1+2\psi)}\dfrac{\partial\psi}{\partial t}\right]}{\left(\dfrac{1}{12P_r}+\dfrac{2}{3P_m}\right)} \quad (28)$$

to a high degree of accuracy. When the gas starts to recombine Q_o will become positive, and T_e will be slightly greater than T_r. Before this time and after recombination the converse is true.

d) The Calculation of Q_1 and the Flux of Radiation

The reader may find it helpful to read Lindquist's discussion of the transfer equation. It must be stressed that our approach is different, and does *not* start from Thomas' (1930) special relativistic treatment of the transfer equation even though we do introduce a locally flat reference frame.

In a fixed coordinate system the transfer equation is

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + (\mathbf{l}\cdot\nabla)I_\nu = -\rho_m\kappa_\nu I_\nu + \rho_m\varepsilon_\nu, \quad (29)$$

where the intensity is I_ν, κ_ν and ϵ_ν are the absorption and emission coefficients, and l is a unit vector whose components are the direction cosines. Since the matter is moving relative to the fixed coordinates we must take into account the relativistic transformations for the solid angle, intensity, frequency, time and direction cosines (see Thomas, 1930). When this is done the transfer equation can be solved by inverting the differential operator. The solution is

$$I_\nu = \sum_{n=0}^{\infty} I_\nu^{(n)} = \sum_{n=0}^{\infty}(-1)^n \times\left\{\left[A\left(\frac{1}{c}\frac{\partial}{\partial t} + \mathbf{l}\cdot\nabla\right)\right]^n \frac{\varepsilon_\nu}{\kappa_\nu}\frac{(1-v^2/c^2)^{3/2}}{(1-\mathbf{v}\cdot\mathbf{l}/c)^3}\right\} \quad (30)$$

where

$$A^{-1} = \rho_m\kappa_\nu(1-\mathbf{v}\cdot\mathbf{l}/c)/\sqrt{1-v^2/c^2}. \quad (31)$$

As written this solution is relativistically exact. The functions ϵ_ν and κ_ν are now proper quantities evaluated in the frame at rest with respect to the matter, and v is the velocity of the matter as measured in the fixed coordinate frame.

The rate of absorption of momentum is given by either side of the expression

$$\frac{1}{c}\iint(\rho_m\kappa_\nu I_\nu - \rho_m\varepsilon_\nu)\mathbf{l}\,d\nu\,d\omega = -\frac{1}{c^2}\frac{\partial}{\partial t}\iint I_\nu\mathbf{l}\,d\nu\,d\omega - \frac{1}{c}\nabla\cdot\iint I_\nu\,\mathbf{l}\mathbf{l}\,d\nu\,d\omega. \quad (32)$$

The expression (30) is now to be inserted for I_ν in the right hand side and the various orders (n) yield the terms $Q_1^{(n)}$

The first approximation to Q_1 uses only the $n=0$ term, and it is an easy matter to show that

$$Q_\alpha^{(0)} = T^\beta_{[r]\alpha;\beta} \quad (33)$$

with

$$T^\beta_{[r]\alpha} = P_r[\tfrac{4}{3}U^\beta U_\alpha - \eta^\beta_\alpha]. \quad (34)$$

As expected, $I_\nu^{(0)}$ leads to the stress-energy tensor for disordered radiation, and in this approximation an observer in the frame moving with the matter measures no net flux of radiation. Since (33) is a covariant equation we can immediately evaluate it in the comoving frame, with α equal to 1.

The next approximation $(n=1)$ has been obtained by Ledoux and Walraven (1958) in covariant form keeping terms up to v/c, which is adequate for our purpose. With a slight change in notation we obtain from his equations

$$Q_1^{(1)} = -\frac{1}{c^2}\frac{\partial F_r}{\partial t} - g_{1\alpha}P^{\alpha\beta}_{;\beta} \quad (35)$$

where

$$F_r = -\frac{c}{3\kappa\rho_m}\frac{\partial\epsilon_r}{\partial r}^{*} \quad (36)$$

and

$$P^{\alpha\beta} = -\frac{1}{3c\kappa\rho_m}\left[g^{\alpha\beta}\frac{\partial\varepsilon_r}{\partial t} + v^\alpha g^{\beta\gamma}\varepsilon_{r;\gamma} + v^\beta g^{\alpha\gamma}\varepsilon_{r;\gamma} - \eta_r[g^{\alpha\beta}v^\gamma_{;\gamma} + g^{\alpha\gamma}v^\beta_{;\gamma} + g^{\beta\gamma}v^\alpha_{;\gamma}]\right]. \quad (37)$$

*In the fixed frame dr is a *proper* differential of distance.

The coefficient of radiative viscosity is

$$\eta_r = \frac{4}{15} \frac{\epsilon_r}{c\kappa\rho_m} \quad . \tag{38}$$

Finally we will need the flux of radiation relative to the matter up to and including the $n = 2$ terms in I_ν. This flux is

$$\mathcal{F} = -\frac{c}{3\kappa\rho_m} \frac{\partial\epsilon_r}{\partial r} + \frac{2}{3} \left(\frac{c}{\kappa\rho_m} \right)^2 \frac{\partial^2 \epsilon_r}{\partial t \partial r} \, . \tag{39}$$

The absorption coefficient (κ) used in the computations is a sum of the Rosseland mean coefficients (corrected for stimulated emission) for electron scattering, free-free and bound-free transitions.

e) The Perturbation Equations

We will be concerned with epochs past that time when the cosmic temperature was (say) 10^8 °K. Since the matter field is nonrelativistic, $(\epsilon_m + P_m)/c^2$ may be replaced by ρ_m. From equations (12a with $i = 1$), (13), (33) and (35), one obtains after some straightforward algebra,

$$\begin{aligned} (\rho_m + \tfrac{4}{3}\rho_r)\left[\frac{\partial V_m}{\partial t} + \mathrm{H}V_m\right] = \\ -(\rho_m + \tfrac{4}{3}\rho_r)\frac{1}{S}\frac{\partial\phi}{\partial r} - \frac{1}{S}\frac{\partial}{\partial r}(P_m + P_r) \\ -\frac{V_m}{c^2}\frac{\partial}{\partial t}(P_m + P_r) + \frac{1}{S}\left\{\frac{\partial}{\partial t}\left[\frac{1}{3c\kappa\rho_m}\frac{\partial\epsilon_r}{\partial r}\right]\right. \\ + \frac{\partial}{\partial_r}\left[\frac{1}{3c\kappa\rho_m}\frac{\partial\epsilon_r}{\partial t}\right] \\ \left. + \frac{4}{5}\frac{\epsilon_r}{c\kappa\rho_m}\frac{1}{S}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_m)\right] - 2\mathrm{H}\frac{1}{3c\kappa\rho_m}\frac{\partial\epsilon_r}{\partial r}\right\} . \end{aligned} \tag{40}$$

To avoid confusion in interpreting the above equation, the δr is *not* a proper differential as it is in equation (36). Since we will later linearize this equation, terms such as $V_m \partial V_m/\partial r$ have been neglected. The quantity in the braces arise from $Q_1^{(1)}$. It should be noticed that in all the other terms $Q_1^{(0)}$ simply adds the radiation field directly to the matter field as should be the case, since if we set κ equal to infinity and remove $Q_1^{(1)}$ (and all higher order terms in Q_1), we lock the radiation and matter together. We thus get the additional inertia and isotropic pressure from the radiation.

The differential equation for ρ'_m is obtained by going through the following steps. Equation (40) is first divided by $(\rho_m + \frac{4}{3}\rho_r)$, the divergence of each term is then taken, and equation (10) is used to eliminate $\nabla^2\phi$. Next, introduce perturbed variables. For example, let

$$\rho_m = \bar{\rho}_m (1 + \rho'_m) \tag{41}$$

with similar expressions for P_r, T_e, etc. The continuity equation (13) becomes, when linearized,

$$\frac{1}{S}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 V_m) = -\frac{\partial\rho'_m}{\partial t} \tag{42}$$

and this is now used to eliminate the terms containing V_m. The last step is to let the spatial part of the perturbed variables vary as

$$\frac{1}{kr} e^{\pm ikr} \qquad (i = \sqrt{-1}) \tag{43}$$

with $k = 2\pi S/\lambda$. For a linear flow, if the perturbation starts out sharing the motion of the mean field,

$$\frac{1}{\lambda}\frac{d\lambda}{dt} = \mathrm{H}. \tag{44}$$

It should be realized that k is constant. Now we replace all space derivatives $\partial/\partial r$ by ik (or by $-ik$). When this is done one finds that the radial dependence factors out of the equation (40) yielding the following equation for the time dependence:

$$\begin{aligned} \frac{\partial^2\rho'_m}{\partial t^2} + 2\mathrm{H}\frac{\partial\rho'_m}{\partial t} = 4\pi G(\rho_m\rho'_m + 2\rho_r\rho'_r) \\ -\frac{4\pi^2}{\lambda^2}\frac{1}{(\rho_m + \frac{4}{3}\rho_r)} \\ \times\left[P_m(\rho'_m + T'_e - \mu') + P_r\rho'_r\right] \\ -\frac{1}{(\rho_m + \frac{4}{3}\rho_r)}\frac{1}{c^2}\frac{\partial\rho'_m}{\partial t} \\ \times\left\{P_r\left[\frac{Q_o}{3P_r} - 4\mathrm{H}\right]\right. \\ \left. + \left[\frac{2}{3}\frac{\beta_e}{(1+\psi)(1+2\psi)}\frac{\partial\psi}{\partial t} - \frac{2}{3}\frac{Q_o}{P_m} - 5\mathrm{H}\right]P_m\right\} \\ -\frac{4\pi^2}{\lambda^2}\frac{1}{(\rho_m + \frac{4}{3}\rho_r)}\frac{c}{3\kappa}\frac{\rho_r}{\rho_m}\left[\frac{12}{5}\frac{\partial\rho'_m}{\partial t} - 2\frac{\partial\rho'_r}{\partial t} + (\rho'_m\right. \\ \left. + \kappa')\left(\frac{Q_o}{3P_r} - 4\mathrm{H}\right) + \rho'_r\left(7\mathrm{H} + \frac{1}{\kappa}\frac{\partial\kappa}{\partial t} - \frac{Q_o}{3}\frac{2}{P_r}\right)\right] . \end{aligned} \tag{45}$$

The two energy equations are immediately obtained from equation (12a) with $i = 0$, and they can be written as

$$\frac{\partial \epsilon_r}{\partial t} + \frac{1}{S}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{4}{3}\epsilon_r V'\right)$$

$$+ \frac{1}{S}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{4}{3}\epsilon_r V_m\right) + 4\mathrm{H}\epsilon_r = Q_o \quad (46)$$

and

$$\rho_m \frac{\partial U_m}{\partial t} - \frac{P_m}{\rho_m}\frac{\partial \rho_m}{\partial t} = - Q_o + \nabla \cdot (K \nabla T_e). \quad (47)$$

The velocity V' is $(V_r - V_m)$. Note that the last term on the right side of equation (47) represents heat conduction.*

The coefficient is

$$K = D\, T_e^{5/2} \quad (48)$$

with the numerical value of D taken from Spitzer (1956). It is easily verified that D, involving as it does only mean field quantities, does not change with cosmic time.

When these two energy equations are added, making use of the continuity equation and remembering that V_m is a first order quantity, the energy equation for the total field is obtained:

$$\rho_m \frac{\partial U_m}{\partial t} + \frac{\partial \varepsilon_r}{\partial t} - \left(\frac{P_m}{\rho_m} + \frac{4}{3}\frac{\varepsilon_r}{\rho_m}\right)\frac{\partial \rho_m}{\partial t}$$

$$+ \frac{1}{S}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{4}{3}\varepsilon_r V'\right) = \nabla \cdot (K \nabla T_e). \quad (49)$$

When we identify $\frac{4}{3}\epsilon_r V'$ with the radiation flux relative to the matter, $\mathcal{F}$, and keep the first order term in $\mathcal{F}$ [see equation (39)] we get the Thomas heat equation (Thomas, 1930). It should be sufficient to state that this heat equation can be derived without assuming V_m is a small quantity, but the longer derivation is not needed; we wish only to correctly identify the term $\frac{4}{3}\epsilon_r V'$ as has just been done.

*From the standpoint of rigorously deriving the post-Newtonian equations this inclusion of an additional term seems arbitrary. To date two approaches handling heat conduction within the framework of relativity have been proposed, one due to Eckart (1940) and the other to Landau and Lifshitz (1959). These two treatments give the same result only for sufficiently small rates of heat transfer, and these difficulties can be ascribed to the lack of a satisfactory kinetic theory for a relativistic system of particles. For the problem we are considering, equation (47) is valid as written.

Equation (46) can now be written in terms of the perturbed variables and we get

$$\frac{\partial \rho'_r}{\partial t} = \frac{4}{3}\frac{\partial \rho'_m}{\partial t} - \frac{4\pi^2}{\lambda^2}\left[\frac{c}{3\kappa\rho_m}\rho'_r - \frac{2}{3}\left(\frac{1}{\kappa\rho_m}\right)^2 \frac{\partial \rho'_r}{\partial t}\right]$$

$$+ \frac{Q_o}{3P_r}(Q'_o - \rho'_r). \quad (50)$$

In Appendix A we have calculated Q_o Q'_o from Compton (and inverse Compton) scattering, free-free emission and absorption, and free-bound transitions.

The linearization of equation (47) requires a great deal of algebra and so an outline of these calculations is given in Appendix B. The result is

$$\frac{\partial T'_e}{\partial t} = B_1 \frac{\partial \rho'_m}{\partial t} - B_2 \frac{\partial \rho'_r}{\partial t} - B_3 T'_e - B_4 \rho'_m - B_5 \rho'_r, \quad (51)$$

where the B_i are given by the following expressions:

$$B_1 = \frac{1}{B_6}\left[\frac{2}{3} + (1 + \frac{2}{3}\beta_e)\frac{\psi}{(1+2\psi)(2+\psi)}\right] \quad (52a)$$

$$B_2 = \frac{1}{B_6}\left[\frac{1}{4}(1+\frac{2}{3}\beta_e)(1+\beta_r)\frac{\psi}{(1+2\psi)(2+\psi)}\right] \quad (52b)$$

$$B_3 = \frac{1}{B_6}\left\{\frac{2}{3P_m}\left[Q_o(A-1) + \frac{4\pi^2}{\lambda^2} D\, T^{7/2}\right]\right.$$

$$- \frac{Q_o}{3P_m}\frac{\psi}{(1+2\psi)(2+\psi)}$$

$$- \frac{\psi}{(1+\psi)(1+2\psi)}\left[\frac{2}{3}\beta_e\right.$$

$$\left.\left. - (1+\frac{2}{3}\beta_e)\frac{(1+\psi)(1-\psi^2)}{(1+2\psi)(2+\psi)^2}\right]\frac{1}{\psi}\frac{\partial\psi}{\partial t}\right\} \quad (52c)$$

$$B_4 = \frac{1}{B_6}\left\{\frac{2}{3P_m} Q_o (C-1)\right.$$

$$+ \frac{\psi}{(1+2\psi)(2+\psi)}\left[\frac{2}{3}\frac{Q_o}{P_m} - 2(1+\frac{2}{3}\beta_e)\right.$$

$$\left.\left. \times \frac{(1-\psi^2)}{(1+2\psi)(2+\psi)}\frac{1}{\psi}\frac{\partial\psi}{\partial t}\right]\right\} \quad (52d)$$

$$B_5 = \frac{1}{B_6}\left\{\frac{2}{3P_m} Q_o B\right.$$

$$-\frac{1}{4}(1+\beta_r)\frac{\psi}{(1+2\psi)(2+\psi)}\left[\frac{2}{3}\frac{Q_o}{P_m}\right.$$

$$-2\left(1+\frac{2}{3}\beta_e\right)\frac{(1-\psi^2)}{(1+2\psi)(2+\psi)}\frac{1}{\psi}\frac{\partial\psi}{\partial t}$$

$$\left.\left.-\beta_r\frac{(1+\frac{2}{3}\beta_e)}{(1+\beta_r)}\left(H-\frac{Q_o}{12P_r}\right)\right]\right\} \quad (52e)$$

$$B_6 = 1+\frac{1}{2}\left(1+\frac{2}{3}\beta_e\right)\frac{\psi}{(1+2\psi)(2+\psi)} \quad . \quad (52f)$$

The three equations which describe the development of a perturbation are (45), (50), and (51).

III. THE SOLUTIONS

From now on when we speak of the mass of a perturbation we mean that number obtained from

$$M = \frac{4}{3}\pi\rho_m\left(\frac{\lambda}{2}\right)^3, \quad (53)$$

and when we quote a value for the optical depth we will have used the expression

$$\tau = \kappa\rho_m\lambda = 2\kappa\rho_m^{2/3}\left(\frac{3M}{4\pi}\right)^{1/3} \quad . \quad (54)$$

Before presenting the numerical solutions it will be helpful to establish some general properties of the flow. To illustrate the results of the next paragraph we shall adopt 3° K for T_r, 10^{-30} for ρ_m, 10^{12} $M_\odot$, and a reference time (t_o) specifying the epoch when the gas temperature was 10^8 degrees. For the various cases we will soon consider, t_o is generally around 2.3 x 10^4 seconds (see Table 1). The computed numbers for the example are in parenthesis after the corresponding expressions.

When radiation is dominant over matter a necessary condition for growth is that the wavelength is greater than the critical Jeans length

$$\lambda_{c;r} = \left(\frac{\pi c^2}{8G\rho_r}\right)^{1/2} \quad . \quad (55)$$

Since $\lambda_{c;r} \sim S^2$, $\lambda \sim S$, $S \sim \sqrt{t}$ and $H = 1/(2t)$, we can readily see that $\lambda = \lambda_{c,r}$ when $t = t_1$, where

$$t_1 = t_0\left(\frac{\lambda}{\lambda_{c;r}}\right)^2_{|t_0} \quad (8.3\times10^8 \text{ sec}). \quad (56)$$

If λ equals α times the distance to the particle horizon at time t_2, then

$$t_2 = t_0\left(\frac{H\lambda}{\alpha c}\right)^2_{|t_o}\left\{\begin{array}{ll}\alpha=1: & 2.7\times10^9 \text{ sec.}\\ \alpha=\frac{1}{5}: & 6.8\times10^{10} \text{ sec}\end{array}\right. \quad (57)$$

Evidently the patch comes inside the horizon at around t_2 (α=1) and then enters into phase II development. We may be assured the time derivatives of μ and ν are of no importance at $t_2(\alpha$=0.2). This generally is around the third oscillation maxima. From equation (50) we can equate the photon diffusion time scale to H^{-1} and get

$$t_3 = M^{4/9}\left[\frac{3^{5/3}\kappa\sqrt{t_o}\,\rho_m\,(t_o)^{1/3}}{2^{7/3}\,c\,\pi^{8/3}}\right]^{2/3}(7.5\times10^{11} \text{ sec})$$

(58)

which marks the beginning of phase III. (It is interesting to note that in this example $\tau(t_0)$ equals 6.8×10^9, while $\tau(t_3) = 200$.) Clearly t_3 is well before the beginning of the fourth phase, for when $T \simeq 3000°$ K we find $t \simeq 3\times10^{13}$ sec:

Although the numerical integrations were begun at time t_0 corresponding to $T = 10^8$ °K, we will present the results starting at the later epoch when the patch is within the particle horizon. To ensure the expression for the radiation stress remains valid, all calculations were terminated when τ equaled ten.

The starting values were the following. Because of the powerful mechanism for energy exchange it is just as well to begin with $T'_e = \rho'_r/4$. To ensure a linear flow is not violated we used $\rho'_m(t_0) = 10^{-20}$, and ratios of ρ'_m/ρ'_r equal to 1, 4/3, and 5/3 were tried. Within this range we found it made virtually no difference which value was used, for the development went to the ratio 4/3 after a fairly short time. Two values for $\partial\rho'_m/\partial t$ at t_0 were used. The first was $\partial\rho'_m/\partial t = 0$ and the second was $\partial\rho'_m/\partial t = 3H\rho'_m$. (The latter number means the patch is initially changing at the same rate as the background.) Again there was hardly any noticeable difference resulting from the use of these two numbers. The numerical integrations were performed using the iterative scheme described by Veronis (1963), and we estimate the final errors are around 0.05%.

The following table lists some of the background characteristics at the reference time t_0.

TABLE 1

Some Characteristics of the Mean Field

Present black-body temp. / Present matter density	3 / 10^{-30}	0.3 / 10^{-30}	3 / 10^{-29}	0.3 / 10^{-29}
Value of t_0	2.30×10^4	2.26×10^4	2.30×10^4	1.92×10^4
ρ_m (t_0)	3.7×10^{-8}	3.7×10^{-5}	3.7×10^{-7}	3.7×10^{-4}
ρ_r (t_0)	8.4×10^{-4}	8.4×10^{-4}	8.4×10^{-4}	8.4×10^{-4}
H (t_0)	2.17×10^{-5}	2.22×10^{-5}	2.17×10^{-5}	2.60×10^{-5}
$-\delta\,(t_0) = \lvert (T_e - T_r)/T_e \rvert\,\vert_{t_0}$	8.83×10^{-16}	9.02×10^{-16}	8.83×10^{-16}	1.06×10^{-15}
$\rho_r = \rho_m$ when t equals	9.3×10^{12}	9.3×10^{6}	9.3×10^{10}	9.3×10^{4}

a) Solutions for $\rho_m(P) = 10^{-30}$ and $T_r(P) = 3°$ K.

A 10^7 M$_\odot$ perturbation comes inside the horizon at 10^6 seconds and the subsequent development is shown in the first figure.
Since the oscillation maxima (the solid dots) tend to become crowded, we have drawn an envelope curve (the dashed line) to indicate what happens. During the adiabatic phase the oscillations are symmetric about $\rho'_m = 0$ and so only the positive swings are shown; and of these, only a few are drawn in some detail. As predicted, around time t_3 (10^{10} seconds) the oscillations start to dampen out, and when the radiation field has relaxed to nearly the background value there then commences a slow gravitational growth (the solid line). An extrapolation to 10^{13} seconds indicates an amplitude about a factor of ten lower than the oscillation maxima before the damping.

A 10^{12} M$_\odot$ perturbation comes inside the horizon slightly after 10^9 seconds and its later development is shown in figure 2.
Again the oscillations are damped out and some of the minima are shown by the open circles. After 10^{13} seconds the system has settled down to a slow growth.

It is interesting to see some of the detailed interaction between the two fields. Figure 3 shows the radiation field reaching a maximum value slightly ahead of the matter. Also, the ρ'_m oscillations are rapidly decreasing in amplitude while ρ'_r is steadily decreasing to the background value (during this era ρ'_r is still oscillating with positive and negative values).

We have computed the development for a 10^{15} M$_\odot$ perturbation and the result is what we would now expect. There are only a few oscillations before the calculations end at $\tau = 10$. According to the expression (58) for t_3, the flow would start to become damped around 2.4×10^{13} seconds — but by this time the gas is nearly neutral. Consequently there is little opportunity for damping and the development is essentially adiabatic. We expect little dissipation by the time phase V begins, and since in this latter era the pressure forces are negligible, a gradual growth should then commence.

b) Solutions for $\rho_m(P) = 10^{-30}$ and $T_r(P) = 0.3°$ K.

The general character of the solutions are not much different compared to the 3.0° K cases just discussed. Figure 4 shows the development for a 10^{10} M$_\odot$ perturbation.
When compared with Figure 1 it will be noticed that the oscillation periods are now longer since the sound speed is less. (The wavelengths are nearly the same for these two cases.) The last oscillation is shown in some detail; evidently the damping is quite strong. Since the gas is rapidly becoming neutral when the calculations terminate, we again expect to find a later period of slow growth starting with an amplitude around 10^{-2} smaller than the oscillation maxima during the earlier adiabatic phase.

For M $= 10^{12}$ M$_\odot$ there are a few oscillations beginning at 10^8 seconds, the fourth one occurring around $t = 10^{13}$. A 10^{15} M$_\odot$ perturbation does not oscillate, but the growth is inhibited by the radiation as long as the gas is at least partially ionized. At 10^{12} seconds $\rho_m \sim t^{2/3}$ and the radiation drag is not too important.

c) Solutions for $\rho_m(P) = 10^{-29}$.

A few studies have been performed using this higher density, and to illustrate what happens consider first a 10^{12} M$_\odot$ patch and the (3° K) case. When compared to the corresponding case using $\rho_m(P) = 10^{-30}$, the oscillations occur at times smaller by about 1/3 and the amplitudes are smaller by about a factor of ten until damping begins. Now, however, $\rho_m > \rho_r$ at 10^{11} seconds which is well before recombination, and so the oscillations after this time, say at around 10^{12} seconds, are not driven so strongly by the radiation field. There is still strong damping. When the calculations terminate at 6×10^{12} seconds $\rho_r/\rho_m \simeq 0.1$, $\psi = 0.3$ and it appears that a new period of slow growth will soon begin.

Again, as an illustration, consider $M = 10^{12} M_\odot$ and the (0.3° K) case. Now there are no oscillations. The physical reason is that $\rho_m > \rho_r$ after 10^5 seconds and gravity is dominant over pressure forces. But the radiative stresses are still present and they are of some importance; after well inside the particle horizon (10^9 seconds) ρ'_m is increasing as $t^{0.57}$. The calculations were terminated at 2.3×10^{11} seconds.

Before beginning a study of the non-linear development during the fifth phase, it will be helpful to summarize the results obtained thus far.

If we adopt $\rho_m(P) \sim 10^{-30}$ and $T_r(P) \sim 3°$ K to be reasonably close to the true values then we can conclude that observational uncertainties in these values are not so great as to invalidate the following generalizations. If the mass is around $10^{16} M_\odot$ or more there won't be oscillations prior to recombination, but the rate of growth of ρ'_m will not be rapid. If we write $\rho'_m \sim t^n$ we have never found $n(t)$ as large as unity. For smaller masses there will always be oscillations but they become damped out and this phase is followed by a slow gravitational growth. We have found the amplitude after damping to be about two orders of magnitude smaller than the oscillation maxima during the earlier adiabatic era.

In all the cases studied (i.e., $M > 10^6 M_\odot$), heat conduction was of no importance. Also, since the exchange of energy between the radiation and matter is so efficient, and since the specific heat of the photon field is so large, the perturbed energy transfer $(Q_o Q'_o)$ hardly affected the results. Clearly $Q_o Q'_o$ will be more important for the smaller value of $T_r(P)$. If $T_r(P) \sim 0.1°$ K, it is possible that this energy exchange could be important when the matter starts to become dominant over radiation in density and mechanical pressure.

A number of people have speculated that an interesting instability might occur during the recombination period caused by a lower ionization state and hence lower pressure in the patch. Only the (0.3°) cases studied ran through the recombination era, and no evidence of an increased instability was noticed. But this is not surprising in view of the form of ψ' (equation A12) and also in view of the time it takes an acoustic wave to travel through the patch compared to the time it takes the gas to recombine.

Will a different chemical composition make much difference? Since electron scattering is dominant at time t_3, if there were as much as 25% helium by mass, κ would change by 13% — surely not enough to significantly affect the flow.

IV. THE DEVELOPMENT AFTER RECOMBINATION

During this fifth phase the flow will be adiabatic and Newtonian equations may be used. The objective now is to get the patch to reverse its motion and start collapsing, since rapid release of gravitational energy is required for the fragmentation process leading to a final stellar system. We first calculate the time required to reverse the motion as a function of the perturbation amplitude in density and velocity.

Suppose the radius of the patch is much larger than the Jeans length

$$\lambda_{m;J} = \left(\frac{5}{3}\frac{\pi k T}{G\rho_m m_H}\right)^{1/2} \tag{59}$$

computed for the matter field alone, which allows us to neglect any consideration of pressure. Then the total energy (W) remains constant and is the sum of the kinetic energy (E_k) and gravitational energy (V), where

$$E_k = \int \frac{1}{2} v^2\, dm = \frac{1}{2}\left(\frac{\mathcal{V}}{R}\right)^2 \frac{3}{5} M R^2 C_I \tag{60}$$

and

$$V = -\frac{3}{5} G \frac{M^2}{R} C_V. \tag{61}$$

Also we may write

$$M = \frac{4}{3}\pi\, \rho_P R^3 C_M \tag{62}$$

where C_I, C_V and C_M are dimensionless form factors, $\mathcal{V}$ is the velocity of the patch at the boundary radius R, and ρ_P is the central density of the patch. If we let

$$\rho(r) = \rho_P\left[1 - \frac{r^2}{R^2}\epsilon_P\right], \tag{63}$$

$$\mathcal{V} = r\mathrm{H}\,(1 - \epsilon_v), \tag{64}$$

then at time t_4 when we mark the beginning of this development we find

$$\left(\frac{5}{3}\frac{W}{M}\right) = -\frac{4}{3}\pi G\,(\rho_P R^2)\,_{|t_4}\,\epsilon' \qquad (65)$$

where

$$\epsilon' = 2\epsilon_v + \frac{4}{7}\epsilon_\rho. \qquad (66)$$

In obtaining this result we used $H^2 = \frac{8}{3}\pi G\rho_c$ with ρ_c being the cosmic density.

The energy equation can be rewritten as

$$\mathscr{V}^2 = \left[\frac{H^2R^2}{(1-\epsilon_\rho)}\right]_{t_4}\frac{1}{C_I}\left[C_V\,(1-\frac{3}{5}\epsilon_\rho)\,\frac{R\,(t_4)}{R\,(t)} - \epsilon'\right]. \qquad (67)$$

When the motion is reversed at time t_5,

$$R(t_4)/R(t_5) = \epsilon'\Big/\left[C_V(1-\frac{3}{5}\epsilon_\rho)\right], \qquad (68)$$

which we use to integrate for the reversal time:

$$\Delta t = t_5 - t_4 = \int_{R(t_4)}^{R(t_5)} \frac{dR}{\mathscr{V}} = \frac{\sqrt{1-\epsilon_\rho}\sqrt{C_I}}{\epsilon' H(t_4)}$$

$$\left\{\left[C_V(1-\frac{3}{5}\epsilon_\rho) - \epsilon'\right]^{1/2} \right. \qquad (69)$$

$$\left. + C_V\frac{(1-\frac{3}{5}\epsilon_\rho)}{\sqrt{\epsilon'}}\tan^{-1}\left[\frac{C_V(1-\frac{3}{5}\epsilon_\rho)-\epsilon'}{\epsilon'}\right]^{1/2}\right\}.$$

Since $C_V \geqslant 1$ and ϵ_ρ and ϵ' are small, the last term in the braces is dominant. It is an easy matter to show that the product $C_V\sqrt{C_I}$ is essentially independent of central concentration by using the density distribution

$$\rho\,(r) = \rho_P\, e^{-\frac{r^2}{R^2}\xi^2} \qquad (70)$$

for letting ξ^2 vary from zero to infinity causes $C_V\sqrt{C_I}$ to change from 1.0 to 1.05. Thus we find, regardless of central concentration,

$$\Delta t = \frac{\pi}{2H\,(t_4)}(\epsilon')^{-3/2}. \qquad (71)$$

To illustrate, pick $t_4 = 1.5\times10^{13}$ seconds.* Then $\rho_m(t_4) = 1.4\times10^{-21}$ and

$$\Delta t = 1.3\times10^9 \text{ years if } \epsilon' = 10^{-2}, \qquad (72a)$$

$$\Delta t = 4.1\times10^{10} \text{ years if } \epsilon' = 10^{-3}. \qquad (72b)$$

The continuity equation connects ϵ_v and ϵ_ρ together; evidently we must have perturbations not much less than 1% in amplitude to obtain galaxies.

Continuing with this example, at the reversal time the cosmic density is 4.6×10^{-28}. Using the density distribution (70) enables C_M, C_V, and C_I to be calculated as functions of $\xi^2(t)$. Then a consistent solution gives $\xi^2=6.5$ at reversal, and so the mean density in the patch turns out to be 2.4×10^{-26}. The central density is about 650 times larger than background. For a spherical galaxy, if the mean density in the final system turns out to be around 10^{-24}, a collapse after reversal is required — but not by a large factor, certainly not more than a factor of three. It is clear that this collapse will occur very rapidly and the total time of formation would be less than a few 10^9 years. However, during this collapse the gas will be heated by compression and a study of this later development requires a more elaborate treatment.

V. CONCLUDING REMARKS

The following remarks are supplementary to those at the end of section III.

The amplitude requirement found in the last section can now be traced back to the earlier time t_2 ($\alpha=1$), since there is no evidence for any enhanced instability during recombination. This means we need amplitude around 1% after a perturbation has entered its particle horizon.

As we have mentioned before, at the earlier time when the patch extends outside the horizon, one cannot uniquely determine how the central patch density is changing relative to background. It would thus seem that the problem of the origin of the initial fluctuations and their development during phase I is unsolvable in the sense that a coordinate independent solution does not exist. We must assume, *ab-initio,* an irregular or turbulent state of the cosmic gas.

One final comment concerns a recent article by Joseph Silk (1968). At first sight it would seem Silk's study and the one presented here are duplicates. However, the results and conclusions do differ, the reason being that Silk has studied an unperturbed flow which is *not* oscillating. But as originally stated in our Introduction, an oscillation is inevitable during the second and part of the third phase. This unrealistic unperturbed flow also

*From now on all numerical examples will be based on the values 10^{-30} and 3° K for the present mass density and radiation temperature of the smoothed out universe.

accounts for the difference between this work and the studies by Peebles (1965) and Saslaw (1967).

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This manuscript has been read in draft form by Drs. Bardeen and Woltjer, and I am very grateful for their suggestions which have improved the presentation of this material.

APPENDIX A

THE CALCULATION OF $Q_o Q'_o$

For conditions near to equilibrium,

$$Q_o = E\delta = E(1 - T_r/T_e) \qquad \text{(A1)}$$

and it follows that

$$Q_o Q'_o = Q_o E' + E(T'_e - \frac{1}{4}\rho'_r) \qquad \text{(A2)}$$

to first order in the perturbed quantities.

The three main contributions to E are Compton (and inverse Compton) scattering, free-free emission and absorption, and free-bound transitions. In our calculations the bound-bound transitions may safely be ignored.

From Weymann (1966) the contribution from Compton scattering (c.s.) is

$$E_{\text{c.s.}} = G_{\text{c.s.}} \left(\frac{\psi}{1+\psi}\right) \rho_r \rho_m T_e \qquad \text{(A3)}$$

where

$$G_{\text{c.s.}} = 4\sigma ck/(m_H m_e) \qquad \text{(A4)}$$

*With partial support under AF 49(638)-1358.

and σ is the Thompson scattering cross section.

For the free-free transitions we may write

$$4\pi \int \kappa'_\nu \left[B_\nu(T_e) - B_\nu(T_r) \right] d\nu \qquad \text{(A5)}$$

for the net energy exchange. B_ν is the Planck function and κ'_ν is the free-free absorption coefficient corrected for stimulated emissions (Allen 1963). After some algebra it is found that

$$E_{ff} = G_{ff} \left(\frac{\psi}{1+\psi}\right)^2 \rho_m^2 \sqrt{T_e} \qquad \text{(A6)}$$

where

$$G_{ff} = \left(\frac{\pi^2}{6} - 1\right) \frac{1}{(m_H)^2} \left[\left(\frac{2\pi k}{3m_e}\right)^{1/2} \frac{2^5 \pi e^6}{3hm_e c^3}\right]. \qquad \text{(A7)}$$

The contribution to E from bound-free transitions can be easily derived from formulae given by Aller (1956). For our purposes we need consider only the transitions between the ground state and the continuum. The result is

$$E_{bf} = G_{bf} \left(\frac{\psi}{1+\psi}\right)^2 \rho_m^2 / \sqrt{T_e} \qquad \text{(A8)}$$

where

$$G_{bf} = \frac{k}{(m_H)^2} \mathscr{K}; \mathscr{K} = 3.260 \times 10^{-6} \qquad \text{(A9)}$$

The expression for E' can now be immediately written down:

$$E' = \frac{1}{E} \left\{ \frac{\psi'}{(1+\psi)} \left[E_{\text{c.s.}} + 2\,(E_{ff} + E_{fb}) \right] + E_{\text{c.s.}}\,(\rho'_r + \rho'_m + T'_e) \qquad \text{(A10)} \right.$$

$$\left. + E_{ff}(2\rho'_m + \frac{1}{2} T'_e) + E_{fb}\,(2\rho'_m - \frac{1}{2} T'_e) \right\}.$$

To obtain ψ' we will use Saha's equation modified to read

$$\frac{\psi^2}{(1+\psi)} = \frac{m_H}{h^3} (2\pi\, m_e\, k)^{3/2} \frac{\sqrt{T_e T_r}}{\rho_m} e^{-\chi_H/kT_r}. \qquad \text{(A11)}$$

The above modification involving T_e and T_r is well known (see Strongren, 1948). Since none of the calculations are carried to the point where ψ is less than 10^{-3}, we always have $T_e = T_r$ in the above equation when applied to the mean-field. We therefore obtain for ψ',

$$\psi' = \left(\frac{1+\psi}{2+\psi}\right)\left[\frac{1}{2}\,T'_e + \frac{1}{4}(1+\beta_r)\,\rho'_r - \rho'_m\right] \tag{A12}$$

where $\beta_r = \chi_H/kT_r$.

The final result is

$$Q_o Q'_o = Q_o\,(A\,T'_e + B\,\rho'_r + C\,\rho'_m), \tag{A13}$$

with

$$A = \frac{E}{Q_o} + \frac{1}{E}\left\{\frac{1}{2}\,\frac{1}{(2+\psi)}\left[E_{c.s.} + 2(E_{ff} + E_{fb})\right] + E_{c.s.} + \frac{1}{2}(E_{ff} - E_{fb})\right\}, \tag{A14a}$$

$$B = -\frac{E}{4Q_o} + \frac{1}{E}\left\{\frac{1}{4}\,\frac{1}{(2+\psi)}\left[E_{c.s.} + 2(E_{ff} + E_{fb})\right](1+\beta_r) + E_{c.s.}\right\}, \tag{A14b}$$

and

$$C = \frac{1}{E}\left(\frac{1+\psi}{2+\psi}\right)\left[E_{c.s.} + 2(E_{ff} + E_{fb})\right]. \tag{A14c}$$

APPENDIX B

THE HEAT EQUATION FOR THE MATTER

The following functions are needed to obtain the first order terms in equation (47), and are all easily derived. From the definition of μ (equation 21) and the expression for ψ' written in Appendix A, it follows that

$$\frac{\partial \psi'}{\partial t} = \frac{(1+\psi)}{(2+\psi)}\left[\frac{1}{2}\frac{\partial T'_e}{\partial t} + \frac{1}{4}(1+\beta_r)\frac{\partial \rho'_r}{\partial t} - \frac{\partial \rho'_m}{\partial t} - \frac{1}{4}\beta_r\frac{1}{T_r}\frac{\partial T_r}{\partial t}\rho'_r\right] + \frac{1}{(2+\psi)^2}\frac{\partial \psi}{\partial t}\left[\frac{1}{2}T'_e + \frac{1}{4}\rho'_r(1+\beta_r) - \rho'_m\right]. \tag{B1}$$

We also will make use of the identity

$$\frac{P_m}{\rho_m}\mu\,\beta_e = \chi_H/m_H. \tag{B2}$$

Now the expression (18) is used to obtain

$$\begin{aligned}
\frac{\partial U_m}{\partial t} = \overline{\left(\frac{\partial U_m}{\partial t}\right)} &+ \frac{3}{2}\frac{P_m}{\rho_m}\Bigg\{\left(\frac{1}{T_e}\frac{\partial T_e}{\partial t}\right)T'_e + \frac{\psi}{(1+2\psi)\,(2+\psi)}\left[\frac{1}{2}T'_e + \frac{1}{4}\rho'_r\,(1+\beta_r) - \rho'_m\right]\left(\frac{1}{T_e}\frac{\partial T_e}{\partial t}\right) - \left(\frac{1}{\mu}\frac{\partial \mu}{\partial t}\right)T'_e \\
&- \frac{\psi}{(1+2\psi)\,(2+\psi)} \times \left[\frac{1}{2}T'_e + \frac{1}{4}\rho'_r\,(1+\beta_r) - \rho'_m\right]\left(\frac{1}{\mu}\frac{\partial \mu}{\partial t}\right)\Bigg\} \\
&+ \frac{3}{2}\frac{P_m}{\rho_m}\left(1+\frac{2}{3}\beta_e\right) \times \frac{\psi}{(1+\psi)\,(1+2\psi)}\Bigg\{\left(\frac{1+\psi}{2+\psi}\right)\left[\frac{1}{2}\frac{\partial T'_e}{\partial t} + \frac{1}{4}(1+\beta_r)\frac{\partial \rho'_r}{\partial t} - \frac{\partial \rho'_m}{\partial t} - \frac{1}{4}\beta_r\left(\frac{1}{T_r}\frac{\partial T_r}{\partial t}\right)\rho'_r\right] \\
&+ \frac{1}{(2+\psi)^2}\frac{\partial \psi}{\partial t}\left[\frac{1}{2}T'_e + \frac{1}{4}\rho'_r\,(1+\beta_r) - \rho'_m\right]\Bigg\} + \frac{3}{2}\frac{P_m}{\rho_m}\frac{\partial T'_e}{\partial t} \\
&+ \frac{3}{2}\frac{P_m}{\rho_m}\frac{1}{(1+\psi)\,(2+\psi)\,(1+2\psi)}\frac{\partial \psi}{\partial t}\left\{\left[\left(\frac{1-2\psi^2}{1+2\psi}\right) + \frac{2}{3}\beta_e(1-\psi)\right]\left[\frac{1}{2}T'_e + \frac{1}{4}\rho'_r\,(1+\beta_r) - \rho'_m\right]\right\}.
\end{aligned} \tag{B3}$$

After multiplying by $\rho_m = \overline{\rho}_m\,(1+\rho'_m)$, substituting the appropriate mean-field derivatives and using

$$\frac{1}{\mu}\frac{\partial \mu}{\partial t} = -\frac{\psi}{(1+\psi)\,(1+2\psi)}\left(\frac{1}{\psi}\frac{\partial \psi}{\partial t}\right), \tag{B4}$$

there results after a great deal of algebra,

$$\rho_m \frac{\partial U_m}{\partial t} = \left(\rho_m \frac{\partial U_m}{\partial t}\right) + \frac{3}{2} P_m \left[1 + \frac{1}{2}\left(1+\frac{2}{3}\beta_e\right) \frac{\psi}{(1+2\psi)\,(2+\psi)} \right] \frac{\partial T'_e}{\partial t}$$

$$+ \frac{3}{8} P_m \left(1+\frac{2}{3}\beta_e\right)(1+\beta_r) \frac{\psi}{(1+2\psi)\,(2+\psi)} \frac{\partial \rho'_r}{\partial t}$$

$$- \frac{3}{2} P_m \left(1+\frac{2}{3}\beta_e\right) \times \frac{\psi}{(1+2\psi)\,(2+\psi)} \frac{\partial \rho'_m}{\partial t}$$

$$- \frac{3}{2} P_m \left\{ \left[1 + \frac{1}{2} \frac{\psi}{(1+2\psi)\,(2+\psi)} \right] \left(2\mathrm{H} + \frac{2}{3}\frac{Q_o}{P_m} \right) + \frac{\psi}{(1+\psi)\,(1+2\psi)} \left[\frac{2}{3}\beta_e - \left(1+\frac{2}{3}\beta_e\right) \frac{(1+\psi)\,(1-\psi^2)}{(2+\psi)^2\,(1+2\psi)} \right] \left(\frac{1}{\psi}\frac{\partial \psi}{\partial t} \right) \right\} T'_e$$

$$- \frac{3}{8} P_m (1+\beta_r) \frac{\psi}{(1+2\psi)\,(2+\psi)} \left[2\mathrm{H} + \frac{2}{3}\frac{Q_o}{P_m} - 2\left(1+\frac{2}{3}\beta_e\right) \times \frac{(1-\psi^2)}{(1+2\psi)\,(2+\psi)} \left(\frac{1}{\psi}\frac{\partial \psi}{\partial t} \right) - \beta_r \frac{\left(1+\frac{2}{3}\beta_e\right)}{(1+\beta_r)} \left(\mathrm{H} - \frac{Q_o}{12 P_r} \right) \right] \rho'_r$$

$$+ \frac{2}{3} P_m \left\{ \frac{\psi}{(2+\psi)\,(1+2\psi)} \left[2\mathrm{H} + \frac{2}{3}\frac{Q_o}{P_m} - 2\left(1+\frac{2}{3}\beta_e\right) \frac{(1-\psi^2)}{(1+2\psi)\,(2+\psi)} \times \left(\frac{1}{\psi}\frac{\partial \psi}{\partial t} \right) \right] - \left(2\mathrm{H} + \frac{2}{3}\frac{Q_o}{P_m} \right) \right\} \rho'_m. \quad \text{(B5)}$$

Next, it is easily verified that

$$\frac{P_m}{\rho_m}\frac{\partial \rho_m}{\partial t} = \overline{\left(\frac{P_m}{\rho_m}\frac{\partial \rho_m}{\partial t} \right)} + P_m \frac{\partial \rho'_m}{\partial t}$$

$$- 3\mathrm{H} P_m \left\{ \left[1 - \frac{\psi}{(1+2\psi)\,(2+\psi)} \right] \rho'_m + \left[1 + \frac{1}{2} \frac{\psi}{(1+2\psi)\,(2+\psi)} \right] T'_e + \left[\frac{1}{4}(1+\beta_r) \frac{\psi}{(1+2\psi)\,(2+\psi)} \right] \rho'_r \right\}. \quad \text{(B6)}$$

It is also readily found that

$$\nabla \cdot (\vec{K}\, \nabla T_e) = - \frac{4\pi^2}{\lambda^2} D\, T_e^{7/2}\, T'_e, \quad \text{(B7)}$$

and with

$$Q_o Q'_o = Q_o \,(A\, T'_e + B\, \rho'_r + C\, \rho'_m) \quad \text{(B8)}$$

(derived in Appendix A), all these expressions (B5, B6, B7, and B8) are then substituted into equation (47) to obtain the result in the text.

REFERENCES

Allen, C. W. 1963, *Astrophysical Quantities* (2nd ed; London: Athlone Press).

Aller, L. H. 1956, *Gaseous Nebulae* (London: Chapman & Hall), Chap. IV.

Bonnor, W. B. 1957, *M.N.*, **117**, 104.

Eckart, C. 1940, *Phys. Rev.*, **58**, 919.

Hawking, S. W. 1966, *Ap. J.*, **145**, 544.

Landau, L. D., and Lifshitz, E. M. 1959, *Fluid Mechanics* (London: Pergamon Press), Chap. XV.

Ledoux, P., and Walraven, T. 1958, *Handbuch der Physik,* Band LI, 442.

Lifshitz, E. M. 1946, *J. Phys. USSR,* **10**, 116.

Lifshitz, E. M. and Khalatnikov, I. M. 1963, *Adv. Phys.*, **12**, 185.

Lindquist, R. W. 1966, *Annals of Physics,* **37**, 487.

Peebles, P. J. E. 1965, *Ap. J.*, **142**, 1317.

Sachs, R. K., and Wolfe, A. M. 1967, *Ap. J.* **147**, 73.

Saslaw, W. C. 1967, *M.N.*, **136**, 39.

Silk, J. 1968, *Ap. J.*, No. 2, Part 1, **151**, 459.

Spitzer, L. 1956, *Physics of Fully Ionized Gases* (New York: Interscience).

Strömgren, B. 1948, *Ap. J.*, **108**, 242.

Thomas, L. H. 1930, *Q. Jr. Math.*, **1**, 239.

Veronis, G. 1963, *J. Atmos. Sci.*, **20**, No. 6, 577.

Weymann, R. 1966, *Ap. J.*, **145**, 560.

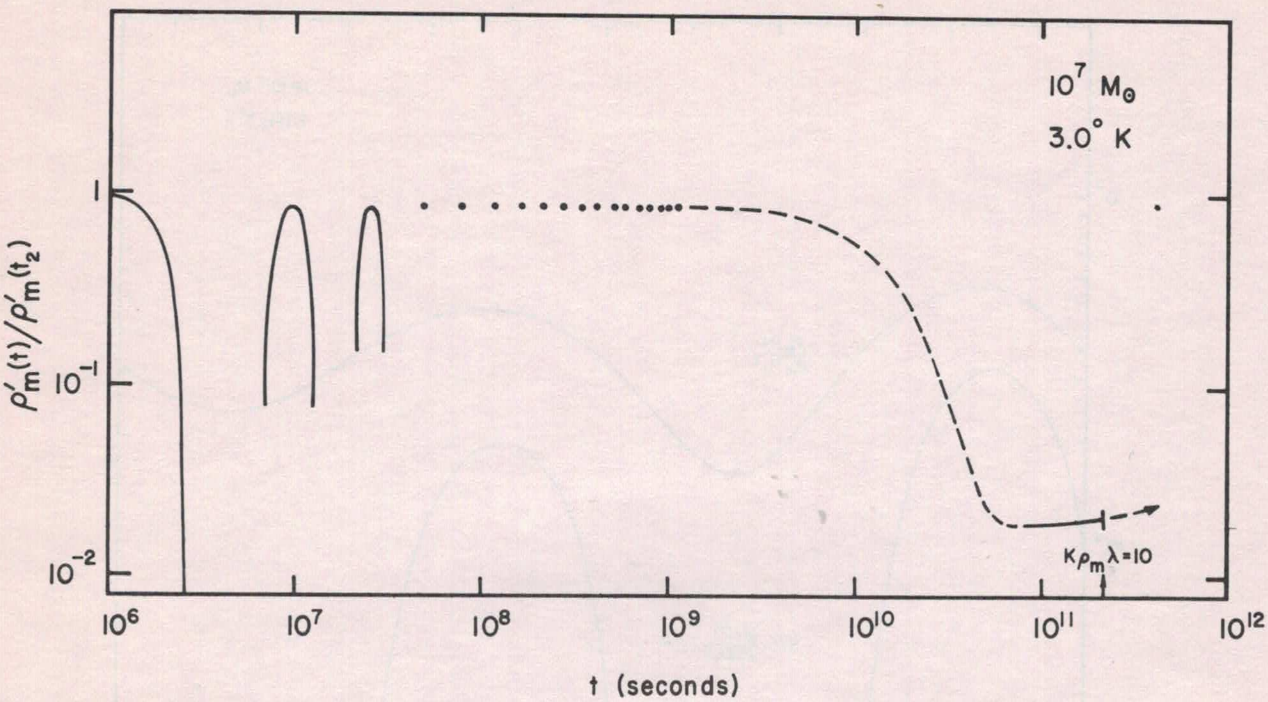

Figure 1: *The development of a 10^7 $M_\odot$ perturbation after it comes inside the particle horizon at time $t_2 = 10^6$ sec. By the third oscillation $\lambda \simeq c/(5H)$, and time derivatives of μ and ν are no longer of any importance. The oscillation envelope (the dashed line) indicates the effect of the radiation and matter fields doing work on each other. In this example the present values taken for the cosmological matter density and radiation temperature are 10^{-30} gm/cm^3 and 3.0° K, respectively.*

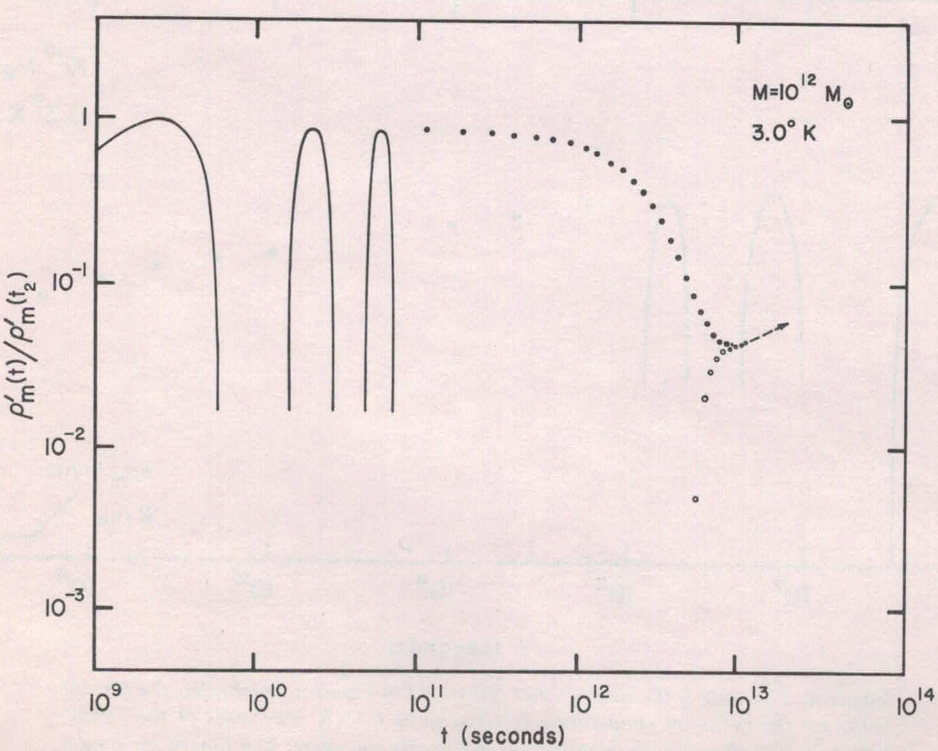

Figure 2: *This case is the same as illustrated in Figure 1, except the mass of the perturbation is 10^{12} $M_\odot$.*

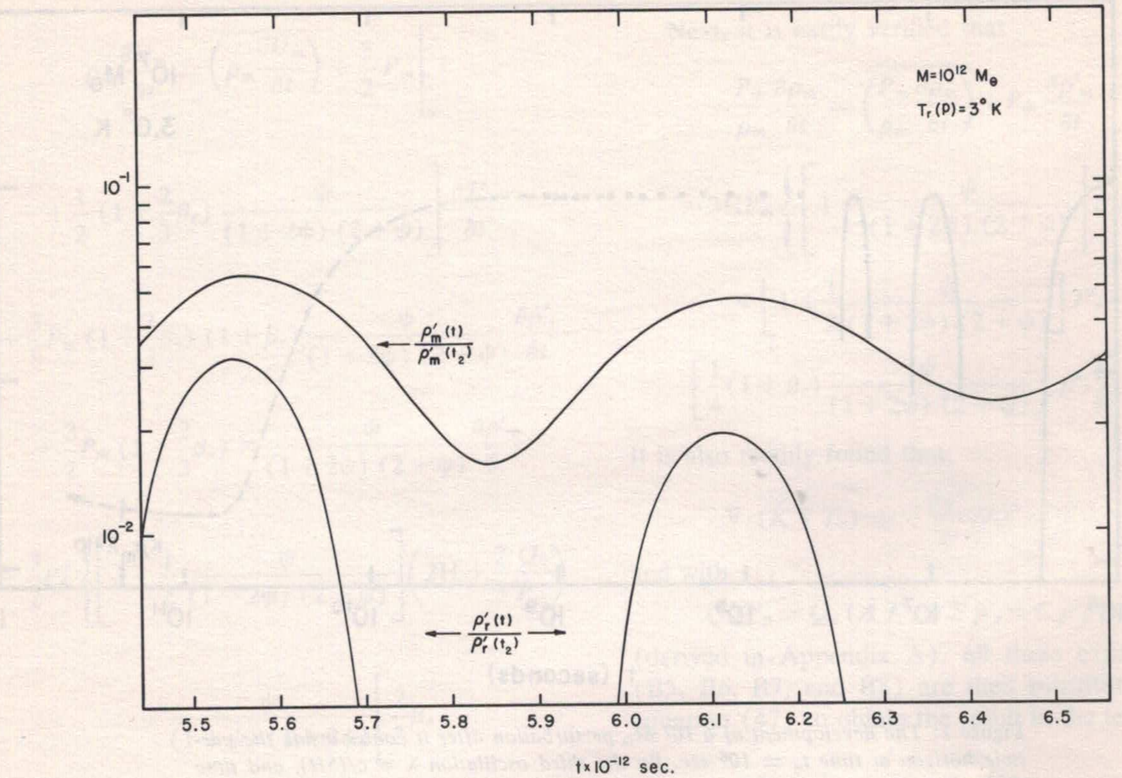

Figure 3: *The detailed development of ρ'_m and ρ'_r around $6x10^{12}$ seconds, for the perturbation whose full development is illustrated in Figure 2.*

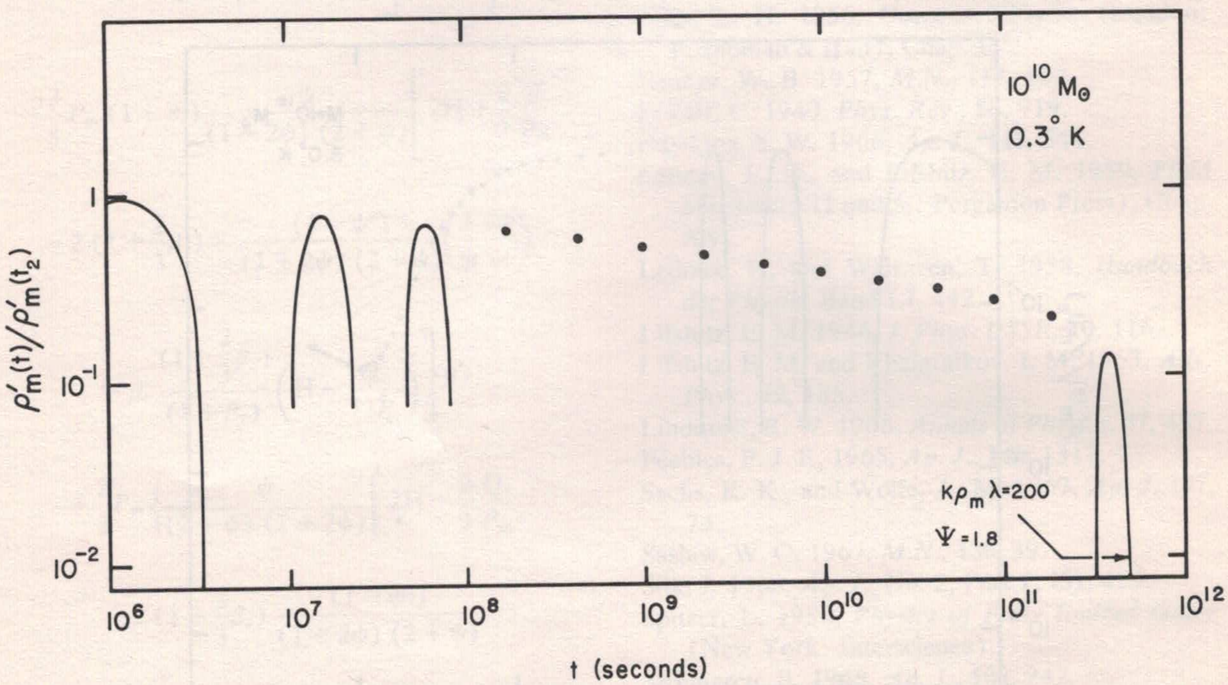

Figure 4: *This case is the same as that illustrated in Figure 1 except that the present value for the radiation temperature is taken to be 0.3° K. The mass of the perturbation is 10^{10} $M_\odot$, and the wavelengths are nearly the same. The longer oscillation period is therefore due to the smaller value of the speed of an acoustic wave.*