Guide Star for SOAR Tip – Tilt Corrector

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■ Introduction

The tip-tilt corrector compensates for motions caused by atmospheric turbulence. The guide channel senses the guide star and feeds a signal to the tip-tilt system, which must zero the signal. This notebook addresses the question, what is the faint limit of guide stars? Our estimates agree with the experience of Glindeman et al, 1997 (PASP 109, 688) with a working system. We conclude:

- At the galactic pole, the chance of finding a guide star within 2 arcmin is 89%.
- The signal sent to the tip-tilt system affects the limiting magnitude. Two possible signals are (1) the normalized flux differences between quadrants of the detector and (2) the centroid of the star. With a detector noise of 3 electrons, flux difference is more sensitive than the centroid by about 0.5–1.5 mag.

■ Quadrant detector

Position is determined by measuring the amount of light in each cell of a quadrant detector. Let the number of photoelectrons in the upper left cell be f_1; upper right, f_2; lower left, f_3; and lower right, f_4. Let the total number in all four quadrants be f. Then the position of the star is

$$x = \mathrm{HWHM}\, h\, (f_1 + f_3 - f_2 - f_4) / f$$

$$y = \mathrm{HWHM}\, h\, (f_1 + f_2 - f_3 - f_4) / f$$

The parameter h depends on the shape of the point-spread function (PSF). For a square, uniform PSF of width FWHM, $h = 1$, which one may verify by moving the star half its width.

The variance of the position is

$$\mathrm{var}(x) = h^2\, \mathrm{HWHM}^2 \left(\frac{4}{f} (u - u^2) + \frac{8 n^2}{f^2} (1 - 2 u + 2 u^2) \right),$$

where $u = (f_1 + f_3) / f$ and $n^2$is the variance of a single quadrant in the absence of light. If the star is centered, the variance in the x-position is

$$\mathrm{var}(x) = h^2\, \mathrm{HWHM}^2 \left(\frac{1}{f} + \frac{4 n^2}{f^2} \right).$$

If in addition the detector noise is negligible, the necessary number of photoelectrons scales as

$$f = h^2\, \mathrm{HWHM}^2 / \mathrm{var}(x)\, .$$

```
nPE::usage = "nPE[errPosn,noiseDetector,h] computes the total number
   of photoelectrons needed for a quadrant detector to determine the x-
   position to errPosn, which is 1-σ error/HWHM.\n noiseDetector is the
   detector noise in one quadrant incurred during one sample.\n h is a
   parameter that depends on the shape of the PSF; its default is 1.";
nPE[errPosn_, noiseDetector_, h_: 1] :=
 Max[e /. NSolve[(4 noiseDetector^2/e^2 + 1/e) == (errPosn/h)^2]]
```

```
? nPE
```

```
nPE[errPosn,noiseDetector,h] computes the total number of photoelectrons needed for a
  quadrant detector to determine the x-position to errPosn, which is 1-σ error/HWHM.
 noiseDetector is the detector noise in one quadrant incurred during one sample.
 h is a parameter that depends on the shape of the PSF; its default is 1.
```

```
mag[x_] := -2.5 Log[10, x]
```

■ Centroid

With many pixels, the centroid is a possible estimate of the position of the star. The centroid is

$$\bar{x} = \frac{1}{f} \Sigma x_i f_i$$

The variance is

$$\mathrm{var}(\bar{x}) = \frac{1}{f^2} \Sigma(x_i^2 - 2 x_i \bar{x} + \bar{x}^2)(n^2 + f_i).$$

Assume the star is centered on the detector. Then

$$\mathrm{var}(\bar{x}) = \frac{1}{f^2} \Sigma x_i^2 (n^2 + f_i).$$

If the detector is an N×N square array with half width W, then

$$\mathrm{var}(\bar{x}) = \frac{1}{f^2} \frac{1}{3} N^2 n^2 W^2 + \frac{1}{f} \frac{1}{2} a^2,$$

where a is the RMS size of the image.

Define $g = (a / \mathrm{HWHM})^2$. Then

$$\mathrm{var}(x) = \frac{1}{f^2} \frac{1}{3} N^2 n^2 W^2 + \frac{1}{f} \frac{g}{2} \mathrm{HWHM}^2,$$

For a gaussian, $g = 1/(2 \log 2)$ and

$$\mathrm{var}(x) = \frac{1}{f^2} \frac{1}{12} N^4 n^2 s^2 + \frac{1}{f} \frac{1}{4 \mathrm{Log} 2} \mathrm{HWHM}^2$$

The variance for the centroid in the absence of detector noise is lower than that of the quadrant method by a factor of 2.8.

```
4 Log@2 // N
```

```
2.77259
```

```
mag[4 Log@2]
```

```
-1.10721
```

For a tip-tilt corrected PSF, the RMS size depends on the maximum radius, because diffraction puts a significant amount of light at large radii. For a motion $\frac{D}{r_0} = 14.19$, which is applicable for the SOAR telescope with $r_0(500\,\mathrm{nm}) = 20$ cm in the R band, $g = 0.943$, 1.153, and 1.328 for $W/\mathrm{HWHM} = 2$, 2.5, and 3 (Loh 2000, seeing10.nb.) Thus is slightly larger than the gaussian value of 0.7213. NB: the HWHM used here is uncorrected value, $\frac{\lambda}{2 r_0}$.

```
nPECentroid::usage =
 "nPECentroid[errPosn,noiseDetector,n,detectorWidth,g] computes the total
   number of photoelectrons needed for an n×n detector to determine the x-
   position to errPosn, which is 1-σ error/HWHM.\n noiseDetector is the
   detector noise in one pixel incurred during one sample.\n detectorWidth
   is the detector width in units of the HWHM.\n h is a quantity that
   depends on the PSF. The default is 1, and it is 0.7213 for a gaussian.";
nPECentroid[errPosn_, noiseDetector_, arraySize_, detectorWidth_, g_: 1] :=
 Max[e /. NSolve[(arraySize^2 noiseDetector^2 detectorWidth^2/(12 e^2) + g/(2 e)) == errPosn^2]]
```

■ Gaussian

I determine h for a gaussian PSF. The PSF is

$$\mathrm{PSF}(x,\ y)\ =\ e^{-(x^2+y^2)/(2\,a^2)}\,/\,(2\,\pi\,a^2).$$

Assume a small shift δ to the right. Then the right half gets an extra amount of light

$$\delta \int_{-\infty}^{\infty} e^{-y^2/(2\,a^2)}\,/\,(2\,\pi\,a^2)\,d\,y$$
$$= \frac{\delta}{a(2\,\pi)^{1/2}}$$

The left half loses that amount. Then

$$f_1 + f_3 - f_2 - f_4 = \frac{2\,\delta}{a(2\,\pi)^{1/2}}$$
$$= \frac{\delta}{\mathrm{HWHM}}\left(\frac{4\log 2}{\pi}\right)^{1/2}$$

since $\mathrm{HWHM} = a(2\log 2)^{1/2}$. Therefore

$$h_{\mathrm{gauss}} = \frac{1}{2}\left(\frac{\pi}{\log 2}\right)^{1/2} = 1.06447$$

and

$$2\ \mathrm{mag}(h_{\mathrm{gauss}}) = -0.14.$$

With a gaussian PSF and in the absence of detector noise, the faintest star is 0.14 mag brighter than for $h = 1$.

```
hGauss = Sqrt[π / (4 Log@2)] // N
```

```
1.06447
```

```
2 mag@%
```

```
-0.135661
```

■ Tip-tilt PSF

A tip-tilt corrected PSF has a concentrated core, which should make h_{TT} smaller. For a motion $\delta = \frac{\lambda}{D}$ with $\frac{D}{r_0} = 14.19$, $h = 0.895$, which is applicable for the SOAR telescope with $r_0(500\,\mathrm{nm}) = 20\,\mathrm{cm}$ in the R band. In the K-band, where $\frac{D}{r_0} = 5.07$, $h = 0.878$. (Loh 2000, seeing10.nb.) Thus h is only slightly smaller. NB: the HWHM used here is uncorrected value, $\frac{\lambda}{2 r_0}$.

```
hTT = 0.895;
```

For the case of negligible detector noise, this makes the limiting magnitude deeper than for *h* = 1 by

```
2 mag@%
```

```
0.240885
```

■ How many counts can be expected from a 0-magnitude star?

Wavelengths of the R, V, and B band centers, h in J*s, c in m/s, energy/photon in V and R

```
r = 7000;
v = 5500;
b = 4400;
h = 6.626 * 10^-34;
c = 3 * 10^8;
```

B-band energy in J.

```
eb = h * c / (b / 10^10)
```

4.51773×10^{-19}

V-band energy in J.

```
ev = h * c / (v / 10^10)
```

3.61418×10^{-19}

R-band energy in J.

```
er = h * c / (r / 10^10)
```

2.83971×10^{-19}

Flux (J/s/m^2/A) of a 0th magnitude star in R, V, and B (Allen, AQ 1973).

```
rpow = 10^(-11.76)
vpow = 10^(-11.42)
bpow = 10^(-11.18)
```

1.7378×10^{-12}

```
General::spell1 : Possible spelling error: new symbol name "vpow" is similar to existing symbol "rpow".
```

3.80189×10^{-12}

```
General::spell : Possible spelling error: new symbol name "bpow" is similar to existing symbols {rpow, vpow}.
```

6.60693×10^{-12}

Light-gathering area of telescope in square meters. This area is the main mirror minus the secondary mirror minus the spider.

$$\textbf{area} = \frac{\pi}{4}\,(4.20^2 - .80^2)$$

```
13.3518
```

Width of R, V, and B filters in Angstroms

```
rwidth = 2200;
vwidth = 900;
bwidth = 1000;
```

```
General::spell1 : Possible spelling error: new symbol name "vwidth" is similar to existing symbol "rwidth".

General::spell :
 Possible spelling error: new symbol name "bwidth" is similar to existing symbols {rwidth, vwidth}.
```

Number of photons in 1s from 0th magnitude star collected by telescope in R and V bands.

```
rphot = rpow * area * rwidth / er
```

1.79757×10^{11}

```
vphot = vpow * area * vwidth / ev
```

```
General::spell1 : Possible spelling error: new symbol name "vphot" is similar to existing symbol "rphot".
```

1.26407×10^{11}

■ Limiting magnitude

The diffraction width in the K-band is

```
2.2*^-6 / 4.25 (180 3600 / π) arcsec
```

```
0.106772 arcsec
```

We assume the positional accuracy to be 1/3 of that for each sample. We assume averaging by the tip-tilt system will make the actual accuracy at 10Hz better by a factor of 3.

The following assumes a 100 Hz frame rate, 0."51 FWHM (typical for SOAR site), a dark current of 250 counts/s (for EG&G AQR-131 APD), a detector QE of 70%, and a 41% loss due to mirrors.   (See http://www.perkinelmer.com/Opto--106/sapdspcm.htm for APD info)

```
res = 0.03 arcsec;
sn = 1;
framerate = 100;
qHWHM = 0.51 / 2 arcsec;
darkCurrentAPD = 250;
qdnoise = Sqrt[darkCurrentAPD / framerate];
qdqe = 0.7;
mirrorqe = 0.9^5;
```

```
mirrorqe qdqe
```

```
0.413343
```

The minimum number of photoelectrons is

$$\texttt{qdcount} = \texttt{nPE}\left[\frac{\texttt{res / sn}}{\texttt{qHWHM}}, \texttt{qdnoise, hTT}\right]$$

```
66.568
```

$$\frac{4\ \texttt{qdnoise}^2}{\texttt{qdcount}}$$

```
0.150222
```

The detector noise is small compared with the photon noise.

The photon rate incident on the telescope is

```
qdrate1 = qdcount * framerate / (mirrorqe * qdqe)
```

```
16104.8
```

Now calculate faintest R and V magnitudes.

```
mag[qdrate1 / rphot]
```

```
17.6193
```

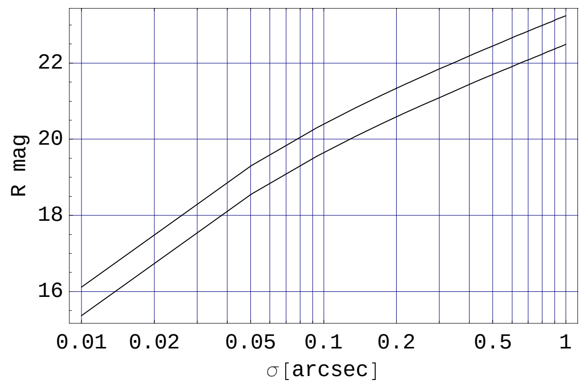
```
mag[qdrate1 / vphot]
```

```
17.237
```

```
Remove@LinearLogPlot
```

```
<< Graphics`Graphics`
```

```
LogLinearPlot[{mag[nPE[x arcsec/qHWHM, qdnoise, hTT] framerate/(mirrorqe*qdqe)/rphot],
  mag[nPE[x arcsec/qHWHM, qdnoise, hTT] (framerate/2)/(mirrorqe*qdqe)/rphot]},
  {x, .01, 1}, Frame → True, FrameLabel → {"σ[arcsec]", "R mag"},
  GridLines → Automatic, PlotRange → All];
```



Caption: Limiting R mag vs. the 1-d positional error σ for a single measurement at 50Hz and at 100Hz. In the K band, $\lambda / D = 0.1$ arcsec. For large σ (0.05 arcsec), the mag changes by 2.5 for every factor of 10 in positional error. For small σ, the mag changes by 5 for every factor of 10 in positional error.

■ Compare with Cecil's results

■ Corrections to GC's result

G. Cecil (December 1999, "SOAR Tip/Tilt Guider, GC99-10") reported 11.1 mag to achieve an measurement of 0.02 arcsec rms at 1kHz.

GC uses

$$x = \frac{\text{arcsec}}{5.1}\,(f_1 + f_3 - f_2 - f_4)/f$$
$$y = \frac{\text{arcsec}}{3}\,(f_1 + f_2 - f_3 - f_4)/f - 0.12\,\text{arcsec}$$
$$\text{var}(x) = (u^{1/2} + (1-u)^{1/2})^2 \Big/ (f/4).$$

(We use the notation defined in the introduction.) He simulates the distribution of u with Zemax. The conversion from ratio of photoelectrons to position is a bit strange; the conversion is different in the two directions and there is an offset in the y-direction.

The formula for variance should be

$$\text{var}(x) + \text{var}(y) = \left(\frac{1}{5.1^2} + \frac{1}{3^2}\right) \text{arcsec}^2\, 4\,(u - u^2)/f.$$

If the star is centered, then the average of the term $u^{1/2} + (1-u)^{1/2}$ is $2^{1/2}$ and

$$(\text{var}(x) + \text{var}(y))_{\text{GC}} / (\text{var}(x) + \text{var}(y))_{\text{Ours}} = 16\left(\frac{1}{5.1^2} + \frac{1}{3^2}\right)^{-1}$$

This changes his magnitude by

```
corr1 = -mag[16 (1/5.1^2 + 1/3^2)^-1]
```

$$\texttt{corr1} = -\texttt{mag}\left[16\left(\frac{1}{5.1^2}+\frac{1}{3^2}\right)^{-1}\right]$$

```
5.07328
```

In the simulation, he does not compensate for tip-tilt, which makes the variance smaller. We estimate the average of this term in order to remove it. Glindemann (1997, PASP 109, 682) finds that the 1-d image motion is $0.41\,(r_0\,/\,D)^{1/6}$ FWHM. For $r_0 = 18$ cm, the RMS motion is about FWHM/4. We assume a flat distribution for u that runs from 1/4 to 3/4. Then the average $< u^{1/2} + (1-u)^{1/2} > \; = \frac{8}{3}\left(\left(\frac{3}{4}\right)^{3/2} - \left(\frac{1}{4}\right)^{3/2}\right)$ is

$$\frac{8}{3}\,(.75^{1.5} - .25^{1.5})$$

```
1.39872
```

This is a factor of

$$\% \,/\, 2^{1/2}$$

```
0.989043
```

lower than the simple estimate $u = \frac{1}{2}$. We will use the simple estimate.

GC used the normalization 1.8×10^{-8}erg/s/cm2/A for a 0-th mag star in the R band, whereas it should be 1.8×10^{-9}. This changes the magnitude by

```
corr2 = mag[10]
```

```
-2.5
```

With all the corrections, GC's magnitude should be

```
mGCcorr = 11.1 + corr1 + corr2
```

```
13.6733
```

■ Our estimate

Assume 0."02 resolution, 1000 Hz frame rate, 0."56 seeing (r_0=18cm), no noise, 55.6% light loss due to mirrors, and 30% loss due to a prism.

Cecil figures the mirror quantum efficiency and extinction directly into telescope area.

```
area2 = Pi * ((420 / 200) ^2 - (80 / 200) ^2 - 0.25 / 10000) * 0.85^4
```

```
6.96967
```

Cecil also uses total light incident on the detector instead of R-band, V-band, etc. I will approximate by calculating the average number of photons incident in BVR bands. The APD QE is a constant 70% across BVR bands, but drops dramatically outside this range. I will ignore U and I bands for this calculation.

```
rphot2 = rpow * area2 * rwidth / er
```

9.38339×10^{10}

```
vphot2 = vpow * area2 * vwidth / ev
```

6.59849×10^{10}

```
bphot2 = bpow * area2 * bwidth / eb
```

1.01928×10^{11}

```
(220 * 0.9 / 2.8 + 90 * 0.8 / 3.6 + 100 * 0.5 / 4.5) / (220 * 0.9 / 2.8)
```

1.43996

```
f0magk5 = % rpow * area2 * rwidth / er
```

1.35117×10^{11}

CG used star SAO 76803 from G. H. Jacoby, D. A. Hunter, & C. A. Christian 1984, ApJS 56, 257. Its dereddened mag is

```
{mag[5.1*^-12 / 1.8*^-9], mag[4.8*^-12 / 3.6*^-9]}
```

{6.36926, 7.18765}

at R and V. In the SAO atlas, V=8.8, and J, H, & C find V=9.06 and the dereddened V=7.31. Thus our estimate matches the dereddened magnitude.

The hwhm at 0.7μ is

$$\text{hwhm} = .7\text{*^-6} \Big/ \left(.18 \left(\frac{.7}{.5}\right)^{6/5}\right) (180\ 3600) / \pi / 2\ \text{arcsec}$$

0.267835 arcsec

and at 0.5μ is

$$.5\text{*^-6} / (.18\ (.5 / .5)^{6/5})\ 180\ 3600\ / \pi / 2\ \text{arcsec}$$

0.286479 arcsec

```
framerate2 = 1000;
qdqe2 = 0.7;
prismqe = 0.7;
```

The 2-d rms resolution is listed as 0."02, so the 1-d resolution is 0."014. There is no detector noise.

$$\text{nPE}[.02 / 2^{1/2}\ \text{arcsec} / \text{hwhm},\ 0,\ \text{hTT}]$$

287.31

```
qdrate2 = % framerate2 / (prismqe * qdqe2)
```

```
586346.
```

```
mag[qdrate2 / f0magk5]
```

```
13.4064
```

```
mGCcorr - %
```

```
0.26689
```

GC's corrected result is 0.3 mag fainter than our estimate.

■ CHARM of Glindemann et al.

Glindemann et al. 1997 (PASP 109, 688) built a tip-tilt system for Calar Alto and found it could use stars as faint as $V = 14$ for these conditions:

FWHM	0."76 in K
1-d error	0."024
sampling	50 Hz
detector noise	6 e-. His detector is an Astrocam CCD.
telescope	3.5m

He gives no Quantum Efficiency, so I will assume 50%. I also assume a light loss per mirror similar to that of SOAR. There are 5 mirrors in Charm and two for the telescope.

The hwhm at 700nm is

$$\text{hwhm} = .76\,(2.2\,/\,.7)^{1/6}\,/\,2\ \text{arcsec}$$

```
0.459908 arcsec
```

$$\text{area3} = \frac{\pi}{4}\,3.5^2$$

```
9.62113
```

Number of photons from 0th magnitude star collected by telescope in R and V bands. We assume Glindemann et al. used a bandpass that is 50% wider than standard V and R.

```
rphot3 = rpow * area3 * rwidth 1.5 / er
vphot3 = vpow * area3 * vwidth 1.5 / ev
```

1.94297×10^{11}

```
General::spell1 : Possible spelling error: new symbol name "vphot3" is similar to existing symbol "rphot3".
```

1.36631×10^{11}

```
framerate3 = 50;
ccdnoise = 6;
ccdqe = 0.5;
mirrorqe3 = 0.9^7
```

```
0.478297
```

We have to make some assumptions on the size of the area read out and the number of pixles. They give an example, which we assume is the case that they actually used. Suppose the 32×32 pixels are binned to read out as a 4×4 array. Suppose the duty cycle is 65%, which is in the middle of their range.

```
nPECentroid[0.024 arcsec / hwhm, ccdnoise, 4, 3.5 arcsec / hwhm]
```

```
1106.33
```

If the binned array is 8×8, then

```
nPECentroid[0.024 arcsec / hwhm, ccdnoise, 8, 3.5 arcsec / hwhm]
```

```
2114.61
```

which is about a factor of 2 brighter.

$$\text{ccdrate1} = \%\% \, \frac{\text{framerate3}}{.65} \Big/ (\text{mirrorqe3} * \text{ccdqe})$$

```
355855.
```

Now calculate faintest R and V magnitudes.

```
rmag3[x_] := -2.5 * Log[10, (x / rphot3)]
```

```
rmag3[ccdrate1]
```

```
14.343
```

```
vmag3[x_] := -2.5 * Log[10, (x / vphot3)]
```

```
General::spell1 : Possible spelling error: new symbol name "vmag3" is similar to existing symbol "rmag3".
```

```
vmag3[ccdrate1]
```

```
13.9607
```

Glindemann et al found V=14th mag. The agreement is perfect.

■ Difference method

They should have used the difference method; the limiting magnitude would have been deeper by

```
nPE[0.024 arcsec / hwhm, ccdnoise, hTT]
```

```
400.032
```

```
nPECentroid[0.024 arcsec / hwhm, ccdnoise, 4, 3.5 arcsec / hwhm]
```

```
1106.33
```

```
mag[%% / %]
```

```
1.10447
```

■ Comparison between Centering Methods: Centroid and Flux Difference

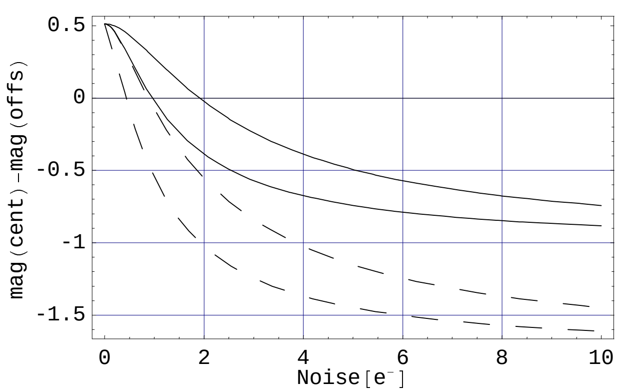
Two measures, the centroid and the normalized flux difference, indicate whether the guide star is centered. Which is better? With a quadrant there is no choice, but either can be used with a CCD. We compare the limiting magnitudes.

These are the conditions: The PSF is a gaussian, and the 1-d positional error is 1/8 of the HWHM. The same detector is used for both methods. The detector covers a width 4 times the HWHM and has N×N pixels. The detector noise is *n* per pixel. With a CCD, the charge is combined without noise to produce 2×2 pixels.

```
dmag[noise_, pixels_, detectorWidth_, err_] :=
 mag[nPECentroid[err, noise, pixels, detectorWidth] / nPE[err, noise, hTT]]
```

```
General::spell1 : Possible spelling error: new symbol name "dmag" is similar to existing symbol "mag".
```

```
Plot[{dmag[n, 4, 4, 1 / 8], dmag[n, 8, 4, 1 / 8], dmag[n, 4, 4, 1 / 16], dmag[n, 8, 4, 1 / 16]},
  {n, 0, 10}, Frame → True, GridLines → Automatic,
  FrameLabel → {"Noise[e⁻]", "mag(cent)-mag(offs)"}, PlotStyle -> {Dashing[{.05, 0}],
    Dashing[{0.05, 0.05}], Dashing[{.05, 0}], Dashing[{0.05, 0.05}]}];
```



Caption: Difference between the limiting magnitude for the centroid method and the flux-difference method vs. detector noise for a 4×4 array (solid lines) and an 8×8 array (dashed lines). The 1-σ error in position is 1/8 HWHM for the upper pair of solid and dashed lines and 1/16 HWHM for the lower pair.

■ Summary

■ Magnitude limit for guide stars

We assume these parameters:

Sampling rate	100 Hz
Spectral response	K star & 400-700 nm
Quantum efficiency	41%
FWHM w/o tip-tilt	0.5 arcsec

Results:

R mag for $\sigma_x = 0.03$ arcsec	17.6
R mag for $\sigma_x = 0.01$ arcsec	15.5

According to a table in Cecil's paper, there is 89% chance of finding a guide star within 2 arcmin at the galactic pole for the fainter magnitude and 48% chance for the brighter one.

■ Comparison with G. Cecil

Cecil's results (December 1999, "SOAR Tip/Tilt Guider, GC99-10") do not agree with ours but can be reconciled when some errors are corrected.

■ Comparison with Glindemann et al.

We can reproduce the limiting magnitude observed by Glindemann et al. 1997, (PASP 109, 688) at the telescope. Glindemann et al. used the centroid method; we find that had they used the difference method their limiting magnitude would have been deeper by 1.1 mag.

■ Comparison between the centering methods

The flux difference is better than the centroid for centering the guide star except when the detector noise is very small. With a detector noise of 3 electrons, flux difference is more sensitive than the centroid by about 0.5–1.5 mag.